Schumpeterian Cycles with Pro–cyclical R&D

Patrick Francois
Department of Economics
University of British Columbia
Vancouver, B.C., Canada.
CEPR and CIAR
francois@interchange.ubc.ca

Huw Lloyd–Ellis
Department of Economics
Queen's University
Kingston, Ontario, Canada
and
CIRPÉE
lloydell@qed.econ.queensu.ca

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Abstract

Recent empirical work finds that R&D expenditures are quite procyclical, even for firms that are not credit-constrained during downturns. This has been taken as strong evidence against Schumpeterian-style theories of business cycles that emphasize the idea that downturns in production may be good times to allocate labor towards innovative activities. Here we argue that the procyclicality of R&D investment is, in fact, quite consistent with at least one of these theories. In our analysis, we emphasize three key features of R&D investment relative to other types of innovative activity: (1) it uses knowledge intensively, (2) it is a long-term investment with uncertain applications and (3) it suffers from diminishing returns over time.

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1 Introduction

Recently, a number of authors have developed theories that revive the Schumpeterian view that economic downturns play a positive role in promoting long run productivity growth.[1] Although there is evidence that some kinds of innovative or restructuring activities are counter–cyclical, most recent empirical studies find that R&D expenditures and employment are pro–cyclical. This is commonly taken as strong evidence against the Schumpeterian view. In this article we demonstrate that, in fact, the pro–cyclical behavior of R&D is quite consistent with a Schumpeterian theory of business cycles. Our analysis emphasizes the fact that R&D is just one part of a multi–stage innovative process through which basic discoveries are eventually translated into productivity improvements. We show that the inherent uncertainty regarding the timing and eventual application of new ideas implies that R&D investment naturally exhibits very different business cycle properties to other forms of innovative activity.

The idea that downturns may induce greater R&D spending is often associated with the impact of negative productivity shocks in "Schumpeterian" endogenous growth models. By lowering wages, negative shocks reduce the opportunity costs of innovative effort and induce higher long term productivity growth (see Aghion and Saint Paul, 1998). A number of recent theories of "endogenous growth cycles" imply the economy alternates between phases of high productivity growth and high fixed capital formation, and phases of low productivity growth but intensive R&D (e.g. Freeman, Hong and Peled (1999), Matsuyama (1999, 2001) and Wälde (2005)). The aim of this literature is to provide an integrated treatment of the economy's secular growth determinants and the source of cyclical fluctuations.[2] However, because the equilibrium growth paths studied in these articles feature no absolute downturns in economic activity and/or are single–sector models, it is difficult to relate these theories to fluctuations at business cycle frequencies. As Christiano and Fitzgerald (1998) illustrate, cycles at frequencies in the 2-10 year range exhibit striking sectoral co-movement in productivity and factor usage through the typical business cycle.

In order to address such business cycle fluctuations, Francois and Lloyd-Ellis (2003) develop a multi-sector model in which expansions and absolute downturns are an intrinsic part of the long–term growth process. Expansions reflect the endogenous clustered implementation of productivity improvements and recessions are the negative side–product of the restructuring that anticipates them.[3] The clustering of implementations is, in turn, driven by the endogenously

[1] "[Depressions] are the means to reconstruct each time the economic system on a more efficient plan. But they inflict losses while they last ..." Schumpeter (1950).

[2] A thorough discussion of the differing approaches that characterize this literature and a preliminary survey can be found at Walde's website: http://www.waelde.com/nv.html

[3] Francois and Lloyd–Ellis (2008) extend the model to allow for capital accumulation and show how fluctuations in the investment rate support the incentives needed to generate the multi–sector cycle.

generated cyclical pattern in demand. In recessions, when demand is low, producers choose to delay implementation until demand improves and thereby generate a higher discounted expected profit stream. This behavior is self-reinforcing in equilibrium: since all implementers delay until such an upturn, their jointly delayed implementation creates the boom in aggregate demand which renders the pattern of delay profitable. Restructuring improvements are essentially the only form of knowledge creation allowed in that model, and they are counter-cyclical.

Most empirical evidence on R&D spending seems to undermine the view that innovation is counter-cyclical. The pro-cyclical behavior of real R&D expenditures, as measured by the NSF, has been documented by many studies, including Griliches (1990), Fatas (2000) and Comin and Gertler (2006).[4] In their study of the cyclical behavior of R&D expenditures by business enterprises in G7 countries, Wälde and Woitek (2004) also find evidence in favour of pro–cyclical R&D spending over the period 1973–2000. The cyclical behavior of R&D has most recently been documented by Barlevy (2007). Using data from both the NSF and Standard & Poor's Compustat database of publicly traded companies, he finds a statistically significant positive correlation between the growth rate of R&D and real GDP growth.[5] Barlevy also finds that the observed aggregate pattern holds at the industry level and that the growth rate in the number of full-time equivalent scientists and engineers employed in industrial R&D closely tracks the growth rate in R&D expenditures.[6]

Figure 1 plots the % deviation from trend in real R&D expenditures and real GDP for the period 1953 to 2003. The index of real R&D expenditures used here is that computed by Barlevy for the Compustat universe. Both series are de-trended using the HP filter (with a scaling parameter of 100 for annual data). One can see immediately that R&D is quite pro-cyclical — the correlation between the two series is 0.5. Over this time period, R&D is about twice as volatile as GDP (the relative standard deviation is 1.9). Most importantly for the analysis here is the fact that R&D tends to be above trend whenever GDP is above trend and vice versa. This correlation at business cycle frequencies in the 2-10 year range is the focus of the present paper.

A number of theories have been advanced to explain why R&D spending is pro–cyclical. Aghion et al. (2005), for example, show how R&D may fall during recessions because of tighter

[4] The NSF define R&D expenditures as those activities whose "purpose is to do one or more of the following things: pursue a planned search for new knowledge, whether or not the research has reference to a specific application; apply existing knowledge to problems involved in the creation of a new product or process, including work required to evaluate possible uses; or apply existing knowledge to problems involved in the improvement of a present product or process."

[5] Compustat defines R&D expenditures as "planned search or critical investigation aimed at discovery of new knowledge" and "translation of research findings or other knowledge to an existing product or process."

[6] These are defined to include "all persons engaged in scientific or engineering work at a level which requires a knowledge of physical or life sciences or engineering or mathematics" and whose "experience is equivalent to completion of a 4-year college course with a major in these fields...".

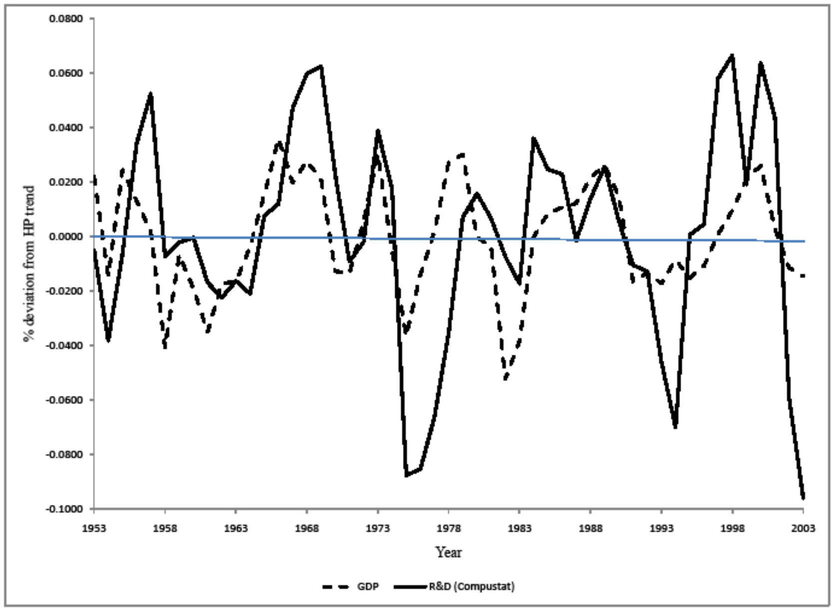


Figure 1: Cyclical component of real GDP and real R&D expenditures by private corporations (Source: Barlevy, 2007)

credit constraints. Although they do find cross–country empirical evidence in support of their theory, Barlevy (2007) finds that R&D is, if anything, more pro-cyclical for those US corporations for whom credit constraints are least likely to bind. He develops a stochastic Schumpeterian growth model in which, although it is socially optimal for R&D to be concentrated during downturns, short–term behavior by innovators results in an inefficiently pro-cyclical allocation of resources to R&D. In a business cycle model with endogenous R&D spending, Comin and Gertler (2006) find that exogenous mark–up shocks can also induce pro–cyclical movements in R&D.[7]

Here, we demonstrate that explicitly introducing R&D into the intrinsic business cycle model of Francois and Lloyd–Ellis (2003), as the first step in a multi–stage innovative process, implies that R&D investment inherently evolves in a pro–cyclical manner.[8] Our explanation does not depend on the existence of tightening credit constraints during downturns nor on short–term behavior by innovators. Moreover, it arises in a model in which both the cyclical process and growth are endogenously determined. Here the pro–cyclical behavior of R&D is the result of three

[7] Harashima (2005) argues that to resolve the pro–cyclical R&D puzzle one must abandon the conjecture that cycles are driven by technology shocks altogether.

[8] In our earlier work, the implications for the cyclical behaviour of R&D was left unspecified.

assumed characteristics: (1) the productivity of inputs into R&D is enhanced by implemented technology, (2) R&D is a long–term investment with uncertain applications and (3) there are diminishing returns to existing knowledge.

Although it is common to assume that ideas discovered through R&D are immediately translated into productivity gains, in reality R&D (as typically defined) is just the first step in the overall innovative process. Kamin, Bijaoui and Horesh (1982), Evangelista et al. (1997) and Baldwin et al. (2004) identify many activities (e.g. product development and design, product specification, prototype construction, manufacturing startup and organizational adjustments) that are crucial for adapting and implementing newly developed technology into production, but which are not generally classified as R&D. In all of these studies, R&D spending accounts for less that 50% of the overall costs of innovation.

In the model developed here, we decompose the innovation process into three distinct stages: R&D, commercialization and implementation. The first two relate to improving on existing production methods, and we model them as distinct.[9] We model "R&D" as a costly process that generates potentially productive ideas whose exact application and timing thereof (if ever) is uncertain. This phase, in itself, does not create something of direct commercial value. We assume that these ideas can be patented immediately, even though their exact application is unknown, so that some share of any eventual return can be reaped by investors. We call this part of the overall innovation process "R&D" because it is the part that is most likely to correspond with the observed measures of R&D used in the empirical literature.

We use the term "commercialization" to refer to the next phase, that in which the process of matching these ideas with particular applications and adapting them for use takes place. Most of this is not likely to be picked up in measured R&D. Commercialization is modelled as a form of costly search by entrepreneurs and/or managers who are motivated by a share of the expected profits.[10] In particular, the rate at which existing ideas are commercialized depends on this entrepreneurial search effort.[11] Once commercially–viable uses have been identified, they can then be implemented in production at an optimally chosen date by licensing to intermediate

[9] This characterization is consistent with the literature on strategic management. For example, according to Kelm, Narayan and Pinches (1995, p. 771) "... there is general agreement that in early phases a firm is involved in attempts to innovate, to find a technical solution to a problem, and that in the later phases firms are involved in attempts at commercialization."

[10] We use entrepreneurs and managers interchangeably because the tasks are performed by the same agents at different phases of the cycle we study. In expansions, when demand is high human capital is mostly devoted to managing production. In contractions when demand falls, its is devoted to more entrepreneurial activities like searching for commercial applications.

[11] "...The function of entrepreneurs is to reform or revolutionize the pattern of production by exploiting an invention or, more generally, an untried technological possibility ... This function does not essentially consist in either inventing anything or otherwise creating the conditions which the enterprise exploits. It consists in getting things done." Schumpeter (1950, p. 132)

goods producers. We denote this the "implementation" phase. The resulting profits are divided between investors in R&D and the entrepreneur/managers according to a simple Nash bargain.

By endogenously treating the R&D phase distinctly from the commercial application phase, and by allowing for the implementation of commercially ready products to be another strategic choice for firms, a recurring pattern emerges. Commercialization is concentrated during downturns, peaking just prior to the subsequent boom. The very fact that this search activity intensifies during recessions implies that the value of ideas whose applications have yet to be determined is maximized at the cyclical peak. After this, the value of these "unmatched ideas" declines temporarily as the likelihood of identifying a commercially viable application before the next expansion falls. Since the expected cost of obtaining each idea does not also fall, R&D actually ceases altogether during recessions, even though commercialization rises.[12] Following an implementation boom, the interest rate rises and the value of unmatched ideas grows as the next phase of intensified commercialization approaches. This induces increased investment in R&D, causing the stock of potentially productive knowledge to rise. Due to diminishing returns to existing knowledge, the equilibrium unit cost of R&D consequently rises with the value of new ideas through the expansion. Thus, the incentives to undertake R&D move in exactly the opposite way over the cycle to those faced by entrepreneur/managers engaged in commercialization.

The remainder of the paper is laid out as follows. Section 2 develops the building blocks of the model. Section 3 solves for the acyclical equilibrium growth path which corresponds to the steady state of the canonical Grossman and Helpman (1991) quality ladders model. Section 4 posits and describes behavior in the cyclical equilibrium. Section 5 elaborates the dynamics over the phases of the cycle and Section 6 derives sufficient conditions for a stationary cyclical equilibrium to exist. Section 7 demonstrates existence of the equilibrium for various sets of parameter values and explores the equilibrium's qualitative characteristics. Section 8 extends the model to allow for unskilled (non-managerial) labor in production. Section 9 discusses the model's implications with respect to the facts of R&D cyclicality and Section 10 concludes.

2 The Basic Model

2.1 Assumptions

There is no aggregate uncertainty. Time is continuous and indexed by $t \geq 0$. The economy is closed and there is no government sector. There are a unit measure of infinitely–lived households

[12] If some of what we have termed "commercialization" is actually picked up in measured R&D, then it will not fall to zero in recessions. However, as we show, even if all of commercialization is attributed to R&D, measured R&D still remains procyclical.

with identical iso–elastic preferences:

$$U(t) = \int_t^\infty e^{-\rho(\tau - t)} \left(\frac{C(\tau)^{1-\sigma} - 1}{1 - \sigma} \right) d\tau \tag{1}$$

where ρ denotes the rate of time preference and σ is the inverse of the elasticity of intertemporal substitution. Each household inelastically supplies 1 unit of skilled human capital and maximizes (1) subject to the intertemporal budget constraint

$$\int_t^\infty e^{-[R(\tau)-R(t)]} C(\tau) d\tau \leq B(t) + \int_t^\infty e^{-[R(\tau)-R(t)]} w(\tau) d\tau \tag{2}$$

where $w(t)$ denotes the wage per unit of human capital, $B(t)$ denotes the household's stock of assets at time t and $R(t)$ denotes the discount factor from time zero to t.

Final output is produced by competitive firms according to a Cobb-Douglas production function utilizing intermediates, x, indexed by i, over the unit interval:

$$Y(t) = \exp \left(\phi t + \int_0^1 \ln x_i(t) di \right). \tag{3}$$

The term ϕ represents constant, exogenous, productivity growth. This plays no essential role in generating the cyclical behavior here, and can be set to zero without affecting qualitative cyclical behavior of the aggregates. We let p_i denote the price of intermediate i. Final output can be used for consumption, $C(t)$, investment in R&D, $I_R(t)$, or (potentially) stored:

$$C(t) + I_R(t) \leq Y(t). \tag{4}$$

Output of intermediate i depends upon the state of technology in sector i, $A_i(t)$, and the human capital, l_i, according to a simple linear technology:

$$x_i^s(t) = A_i(t) l_i(t). \tag{5}$$

There is no imitation, so the dominant entrepreneur in each sector undertakes all production and earns monopoly profits by limit pricing until displaced by a higher productivity rival. We assume that intermediates are completely used up in production, but can be produced and stored for use at a later date. Incumbent intermediate producers must therefore decide whether to sell now, or store and sell later.

Wherever they are ultimately used, new ideas are assumed to dominate old ones by a productivity factor e^γ. This implies that the total potential productivity of the stock of existing knowledge can be expressed as $Z(t) = e^{\phi t + \gamma N(t)}$, where $N(t)$ is the measure of all basic ideas emanating from R&D. We assume that the measure of new ideas emanating from R&D per period is given by

$$\dot{N}(t) = \frac{\mu}{Z(t)} I_R(t), \tag{6}$$

where $\mu > 0$ is a productivity parameter. This specification captures the notion that it becomes more costly to produce each additional idea, the larger the stock of existing ideas. Specifically, one more new idea requires $Z(t)/\mu$ units of investment — effectively there are diminishing returns to existing ideas. However, this effect is offset in the long run by the fact that human capital allocated to R&D becomes more productive as a result of implemented knowledge. Potential productivity therefore evolves through time according to

$$\dot{Z}(t) = \phi Z(t) + \gamma \mu I_R(t), \tag{7}$$

R&D investment is financed by selling claims to households. The expected value of the share of claims to such an idea, accruing to investors in R&D, is denoted by $\Omega(t)$.

In this basic version of the model, we interpret human capital to be that of highly skilled managers and entrepreneurs — those involved in organizing and supervising, as well as innovation. In Section 8, we extend the model to include a second type of labour that is supplied elastically and can be used only in production. As we demonstrate, the model then also generates pro-cyclical fluctuations in employment and wages.

2.2 The Market for Ideas

Although R&D adds to the stock of potentially productive ideas, these ideas are not immediately commercially viable. We model the market for these ideas as a one–sided matching process in which entrepreneur/managers allocate labor effort to searching amongst the stock of potential ideas for those that will be commercially viable in a particular application.[13] The rate of success of this search activity is given by $\delta h_i(t)$, where δ is a parameter, and h_i is the labor effort allocated to search in sector i. At any point in time, entrepreneur/mangers decide whether or not to allocate labor effort to searching for commercially viable ideas, and if they do so, how much labor to devote. The aggregate labor effort allocated to search is given by $H(t) = \int_0^1 h_i(t)dt$. As with R&D, entrepreneurial search is financed by selling claims to households.

Entrepreneurs with commercially–viable innovations must choose whether or not to implement immediately or delay until a later date. Once they implement, the knowledge of how the idea can be made commercially viable becomes publicly available, and can be built upon by rival entrepreneurs. However, prior to implementation, this knowledge is privately held by the entrepreneur. We let $V_i^I(t)$ denote the expected present value of profits from implementing at time t, and $V_i^D(t)$ denote that of delaying implementation from time t until the most profitable time in

[13] Comin and Gertler (2006) and Paterson (2006) use a related two–stage framework to capture the delay between R&D and the adoption of ideas into production. A key difference here is that, once the commerical viability of an idea is identified, there may be a further (strategic) delay before implementation.

the future. It follows that the value of a commercially viable idea is

$$\tilde{V}_i(t) = \max[V_i^D(t), V_i^I(t)] \tag{8}$$

Once an idea is implemented in production, the households who financed R&D receive $(1-\eta)\tilde{V}_i(t)$ and those that financed entrepreneurial search receive $\eta\tilde{V}_i(t)$. As is common in models of search and matching, the share parameter η is treated as an exogenous outcome of a bilateral bargaining process.

It follows that the expected value of a claim to an unmatched idea, $\Omega(t)$, depends on the eventual payoff, the rate at which the idea is matched with an application and the delay before it is implemented. The stock $S(t)$ of "untapped ideas" — ideas emanating from R&D which have not been matched with a particular application — evolves according to

$$\dot{S}(t) = \dot{N}(t) - \delta H(t). \tag{9}$$

We assume that the ideas which constitute this untapped stock are equally likely to be commercially viable. It follows that the rate at which a given idea is matched is given by $\delta H(t)/S(t)$.

In summary, for one of the basic $N(t)$ ideas to increase productivity, and thus influence actual production, it has to be both commercialized, and implemented. Note that we have implicitly imposed the assumption that each idea emanating from R&D has a unique application, so that once it has been matched with a sector, no subsequent matches can occur. This greatly simplifies the exposition, with little loss of generality. The model could be extended to allow for multiple applications without changing the main results.

2.3 Definition of Equilibrium

Given initial state variables $\{A_i(0),\ N(0), S(0)\}$, an equilibrium for this economy consists of:
(1) random processes $\left\{\hat{p}_i(t),\ \hat{x}_i(t).\ \hat{L}_i(t),\ \hat{h}_i(t),\ \hat{A}_i(t),\ \hat{V}_i^I(t),\ \hat{V}_i^D(t)\right\}_{t\in[0,\infty)}$ for each intermediate sector i, and
(2) economy wide sequences $\left\{\hat{Y}(t),\ \hat{w}(t),\ \hat{C}(t), I_R(t),\ \hat{S}(t), \hat{H}(t), \hat{N}(t)\right\}_{t\in[0,\infty)}$
which satisfy the following conditions:
• Households allocate consumption over time to maximize (1) subject to the budget constraint, (2) and the transversality condition.
• Final goods producers choose intermediates, x_i, to minimize costs given prices $\hat{p}_i(t)$. There is free entry into final goods production.
• Each intermediate producer i chooses its price, $\hat{p}_i(t)$, to maximize profits taking into account the demand function of final goods producers, $x_i(p_i)$, and taking the wage and marginal cost of rivals as given.

• The labor market clears

$$\int_0^1 l_i(t)di + H(t) = L. \tag{10}$$

• Arbitrage trading in financial markets implies that, for all assets that are held in strictly positive amounts by households, rates of return must be equal.

• Free entry into commercialization:

$$\delta\eta\tilde{V}_i(t) \leq w(t), \quad h_i(t) \geq 0 \qquad \text{with at least one equality (w.a.l.o.e.)}, \forall\ i. \tag{11}$$

• Free entry into R&D:

$$\mu\Omega(t) \leq Z(t), \quad I_R(t) \geq 0 \quad \text{w.a.l.o.e.} \tag{12}$$

• In periods where there is implementation, entrepreneurs with commercially–viable ideas must prefer to implement rather than delay until a later date.

• In periods where there is no implementation, either there must be no commercially–viable ideas available to implement, or entrepreneurs must prefer to delay rather than implement.

• The aggregate resource constraint (4) must be satisfied.

2.3.1 Direct Implications of Equilibrium

The first–order conditions of the household's optimization require that

$$C(t)^\sigma = C(s)^\sigma e^{R(t)-R(s)-\rho(t-s)} \qquad \forall\ t, s, \tag{13}$$

and that a transversality condition holds: $\lim_{s\to\infty} e^{-R(s)}B(s) = 0$. The derived demand for intermediate i from final goods producers is

$$x_i^d(t) = \frac{Y(t)}{p_i(t)}. \tag{14}$$

The profit maximizing price set by intermediate producer i is given by

$$p_i(t) = \frac{w(t)}{e^{-\gamma}A_i(t)}, \tag{15}$$

and the instantaneous profit earned is

$$\pi_i(t) = (1 - e^{-\gamma})Y(t). \tag{16}$$

Note crucially that firm profits are proportional to aggregate demand. It follows that the total income accruing to labour in production is given by

$$w(t)(1 - H(t)) = e^{-\gamma}Y(t). \tag{17}$$

3 The Acyclical Equilibrium Growth path

There are at least two stationary equilibrium growth paths that are consistent with the conditions described above. Since we are interested in the cyclical properties of this framework, most of the analysis will study the cyclical equilibrium. However, for comparison purposes, it useful to briefly consider the acyclical stationary equilibrium. This closely corresponds to the equilibrium analyzed in the canonical Schumpeterian endogenous growth model of Grossman and Helpman (1991).

Along this path, for which we denote variables by superscript a, all commercially–viable ideas are optimally implemented immediately[14] and aggregates grow at the same constant rate

$$g^a = \delta\gamma H. \tag{18}$$

Consequently, the Euler equation yields a constant interest rate given by

$$r^a = \rho + \sigma(g^a + \phi). \tag{19}$$

Along the balanced growth path, the search no–arbitrage equation must also hold:

$$r^a + \delta H = \frac{\dot{V}^I}{V^I} + \frac{\pi}{V^I}. \tag{20}$$

Free entry into commercialization requires that

$$\eta\delta V^I(t) = w(t) \tag{21}$$

and profits are given by

$$\pi(t) = (e^\gamma - 1)(1 - H)w(t). \tag{22}$$

Substituting into (20) using (18), (21) and (22) yields

$$r^a = \phi + \eta\delta(e^\gamma - 1) - [\eta(e^\gamma - 1) + 1 - \gamma]\,\frac{g^a}{\gamma} \tag{23}$$

Assuming $\gamma < 1 + \eta(e^\gamma - 1)$, this equation yields a negative relationship between r and g. The main reason is that a high steady–state growth rate, g, means that more labor is allocated to search which deflates profits and raises the risk of obsolescence. These tend to offset the positive impact of higher profit growth. Equating (19) and (23) yields the steady state growth rate

$$g = g^a + \phi = \phi + \gamma\frac{\eta(e^\gamma - 1)\delta - \rho + (1 - \sigma)\phi}{\eta(e^\gamma - 1) + 1 - \gamma(1 - \sigma)}. \tag{24}$$

A no–arbitrage equation must also hold for R&D. This is given by

$$r^a = \frac{\delta H}{S}\left[\frac{(1 - \eta)V^I(t) - \Omega(t)}{\Omega(t)}\right] + \frac{\dot{\Omega}(t)}{\Omega(t)}. \tag{25}$$

[14]Since over all time horizons the rate of discount exceeds the rate of growth profits, there is no incentive to delay. This is not always the case in the cyclical equilibrium (see below).

Free entry into R&D requires that

$$\mu\Omega(t) = Z(t) = e^{\phi t + \gamma N(t)}, \tag{26}$$

and the productivity of implemented knowledge can be expressed as

$$\bar{A}(t) = e^{\gamma(N(t)-S(t))}. \tag{27}$$

Substituting into (25), using (18), (21), (26) and (27) and re–arranging yields

$$\left(\frac{r^a}{g^a} - 1\right)\gamma S^a = \frac{(1-\eta)\mu e^{-\gamma} e^{-\gamma S^a}}{\delta\eta} - 1 \tag{28}$$

Given r^a and g^a, this equation pins down the unique steady–state stock of unmatched ideas, S^a. Note that for this equilibrium path to exist, the parameters must be such that $r^a > g$, $g^a > 0$ and $(1-\eta)\mu e^{-\gamma} > \delta\eta$.

Notice that, in this acyclical equilibrium, the R&D sector plays an essential, but largely supportive role: it produces and maintains a sufficiently large stock of knowledge to ensure that the economy grows at the required rate. The productivity of the R&D technology, μ, has no impact on long–run growth. In effect the role is very similar to that of capital accumulation in a standard endogenous growth model. Thus, in this case, "entrepreneurship" (via the commercialization process) is the key driver of growth.

4 The Cyclical Equilibrium Growth Path

Suppose that implementation occurs at discrete dates denoted by T_ν where $v \in \{1, 2, ..., \infty\}$. We adopt the convention that the vth cycle starts at time T_{v-1} and ends at time T_ν. We denote values of variables the instant after implementation by a 0 subscript.[15] Figure 2 illustrates the time line associated with our hypothesized cycle. After implementation at date T_{v-1}, an **expansion** is triggered by a productivity boom and continues through subsequent consumption growth. During this phase, commercialization ceases and consequently all labor effort is used in production. R&D spending is highest during this phase so that the stock of knowledge grows. At some time T_v^*, search commences and labor starts to be withdrawn from production. Commercially viable ideas are not implemented immediately but are withheld until time T_v. During this **contraction** phase, investment in R&D slows and search continues to accelerate in anticipation of the subsequent boom. As aggregate demand falls, labor continues to be released from production into the increased search.

[15] Formally, $X_0(T) = \lim_{t \to T^+} X(t)$.

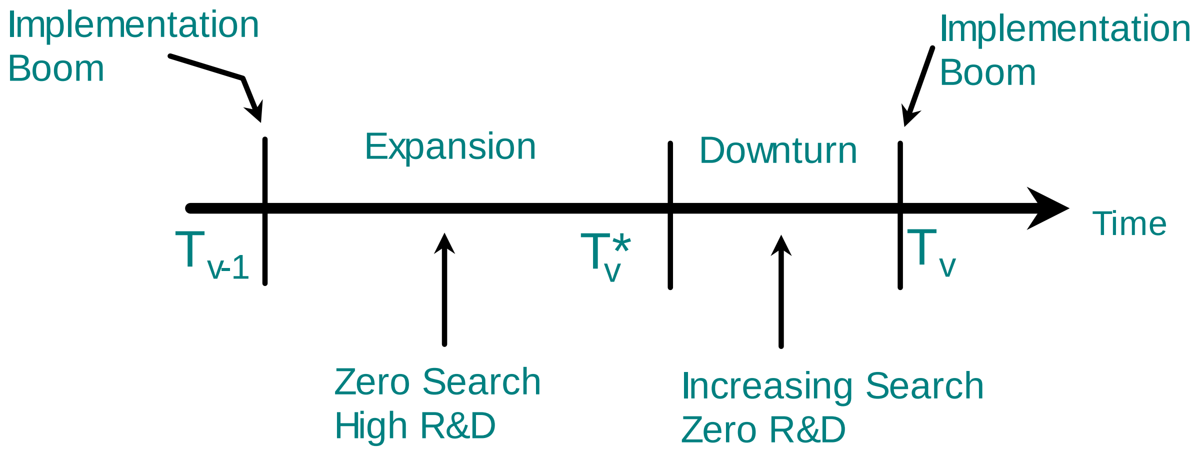


Figure 2: Innovation over the cycle

Let $P_i(s)$ denote the probability that, since time T_{v-1}, no entrepreneurial success has been made in sector i by time s. It follows that the probability of there being no entrepreneurial success at time T_v conditional on there having been none by time t, is given by $P_i(T_v)/P_i(t)$. Hence, the value of an incumbent firm in a sector where no innovation has occurred by time t during the vth cycle can be expressed as

$$V_i^I(t) = \int_t^{T_v} e^{-\int_t^\tau r(s)ds} \pi_i(\tau) d\tau + \frac{P_i(T_v)}{P_i(t)} e^{-\beta(t)} V_{0,i}^I(T_v), \tag{29}$$

where

$$\beta(t) = R_0(T_v) - R(t) \tag{30}$$

denotes the discount factor used to discount from time t during the cycle to the beginning of the next cycle. The first term in (29) represents the discounted profit stream that accrues with certainty during the current cycle, and the second term is the expected discounted value of being an incumbent at the beginning of the next cycle.

Lemma 1 *In a cyclical equilibrium, the identification of commercially viable ideas can be credibly signalled immediately and all search in that sector stops until the next round of implementation.*

If an entrepreneur's announcement is credible, other entrepreneurs will exert their search efforts in sectors where they have a better chance of becoming the dominant entrepreneur. One might imagine that unsuccessful entrepreneurs would have an incentive to mimic successful ones by falsely announcing success to deter others from entering the sector. But there is no advantage to this strategy relative to the alternative of allocating effort to the sector until, with some probability, another entrepreneur is successful, and then switching to another sector.

In the cyclical equilibrium, entrepreneurs' conjectures ensure no more entrepreneurship in a sector once a signal of success has been received, until after the next implementation. The expected value of an entrepreneurial success occurring at some time $t \in (T_v^*, T_v)$ but whose implementation is delayed until time T_v is thus:

$$V_i^D(t) = e^{-\beta(t)} V_{0,i}^I(T_v), \tag{31}$$

Since no implementation occurs during the cycle, the entrepreneur implementing at T_v is assured of incumbency until at least T_{v+1}. Incumbency beyond that time depends on the probability that no commercially viable improvement has been identified in that sector up until then.

The symmetry of sectors implies that innovative effort is allocated evenly over all sectors that have not yet experienced an innovation within the cycle. Thus the probability of not being displaced at the next implementation is

$$P_i(T_v) = \exp\left(-\int_{T_v^*}^{T_v} \delta h_i(\tau) d\tau\right). \tag{32}$$

Given the simplifying assumption that all ideas have equal likelihood of being commercialized, it follows that $\Omega(t)$, the value of a claim to a new idea that has yet to be matched with a particular application must satisfy the Bellman equation, which is identical to that in the acyclical equilibrium, equation (25):

$$r(t)\Omega(t) = \frac{\delta H(t)}{S(t)} \left[(1-\eta)V^D(t) - \Omega(t)\right] + \dot{\Omega}(t). \tag{33}$$

Note that since the probability of being matched is no greater than 1, it must be the case that $(1-\eta)V^D(t) \geq \Omega(t)$.

5 Within–cycle dynamics

Within a cycle, $t \in (T_{v-1}, T_v)$, productivity grows at the exogenous rate ϕ. It follows that the wage also grows at the same rate (see Appendix):

$$w(t) = e^{-\gamma} e^{\phi t} \bar{A}_{v-1}, \tag{34}$$

where

$$\bar{A}_{v-1} = \exp\left(\int_0^1 \ln A_i(T_{v-1}) di\right). \tag{35}$$

Note that the wage is less than its marginal product by a constant factor $e^{-\gamma}$, reflecting the fact that a fraction $1 - e^{-\gamma}$ goes in the form of profits to intermediate producers. Aggregate output can be expressed as

$$Y(t) = e^{\phi t} \bar{A}_{v-1} \left[1 - H(t)\right]. \tag{36}$$

To afford a stationary representation of the economy, we normalize aggregates by dividing by total factor productivity and using lower–case letters to denotes the deflated variables:

$$c(t) = \frac{C(t)}{e^{\phi t}\bar{A}_{v-1}}, \qquad y(t) = \frac{Y(t)}{e^{\phi t}\bar{A}_{v-1}}, \; z(t) = \frac{Z(t)}{e^{\phi t}\bar{A}_{v-1}}. \tag{37}$$

Consequently, the intensive form production function is given by

$$y(t) = 1 - H(t). \tag{38}$$

The household's Euler equation during the cycle can be expressed as

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho}{\sigma} - \phi, \tag{39}$$

where $r(t) = \dot{R}(t)$. The normalized potential productivity evolves according to

$$\frac{\dot{z}(t)}{z(t)} = \frac{\dot{Z}}{Z} - \phi = \mu\gamma\left(\frac{1 - H(t) - c(t)}{z(t)}\right), \tag{40}$$

since $\dot{z} > 0$ implies $\mu\Omega(t) = Z(t)$.

5.1 The Expansion $(T_{v-1} \to T_v^*)$

During the expansion all labour is used in the production of consumption goods and R&D, $H(t) = 0$. From (12) and (33) it follows that

$$\frac{\dot{z}(t)}{z(t)} = \frac{\dot{\Omega}(t)}{\Omega(t)} - \phi = r(t) - \phi. \tag{41}$$

Combing these conditions with (36), (39) and (40) yields the dynamical system:

$$\frac{\dot{c}(t)}{c(t)} = \frac{\mu\gamma}{\sigma}\left(\frac{1 - c(t)}{z(t)}\right) - \frac{\hat{\rho}}{\sigma} \tag{42}$$

$$\frac{\dot{z}(t)}{z(t)} = \mu\gamma\left(\frac{1 - c(t)}{z(t)}\right). \tag{43}$$

where $\hat{\rho} = \rho - (1 - \sigma)\phi$. These dynamics are illustrated using the phase diagram in Figure 3.

In the equilibrium that we study, the economy evolves along a transition path like AB in Figure 3. In the case illustrated, both consumption and the stock of knowledge grow during the expansion. However, if the path extended to the right of the $\dot{c} = 0$ locus, it is also possible that consumption declines towards the end of the phase. In either case, given initial values for consumption and the stock of knowledge, the dynamical system above yields a unique path for $c(t)$ and $z(t)$ at each date t during the expansion. In particular, we can describe the path of consumption as

$$c(t) = F(t;\; c_0(T_{v-1}),\; z_0(T_{v-1})) \qquad \forall\, t \in [T_{v-1}, T_v^*], \tag{44}$$

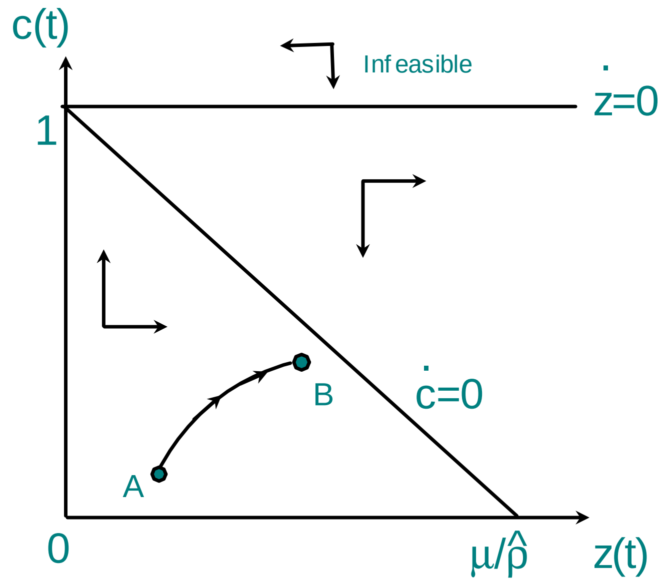


Figure 3: Phase Diagram

where the function $F(t;\cdot)$ is implicitly defined.

As a result of the boom, wages rise rapidly. Since the next implementation boom is some time away, the present value of allocating a unit of labour effort to search falls below the wage, $\delta\eta V^D(t) < w(t)$. During the expansion , the expected value of search, $\delta\eta V^D(t)$, grows at the rate of interest, but continues to be dominated by the wage. As a result of ongoing R&D, the stock of ideas expands. However none of these ideas is matched with a sector, so that

$$\dot{S}(t) = \dot{N}(t). \tag{45}$$

At date T_v^*, $\delta\eta V^D(T_v^*) = w(T_v^*)$ for the first time. If all labor were to remain in production, the returns to search effort would strictly dominate those in production. As a result, labor effort is reallocated from production into search, triggering the next phase of the cycle. The following Lemma demonstrates that during this transition, labor effort shifts rapidly from one activity to the other:

Lemma 2 *: At T_v^*, investment in R&D falls discretely to zero and the human capital devoted to searching for commercially viable ideas jumps discretely to $H_v = H_0(T_v^*) > 0$.*

Although investment in R&D falls discretely at $t = T_v^*$, consumption is constant across the transition between phases because the discount factor does not change discretely. It follows that the decline in output due to the fall in R&D investment demand must be proportional to the human capital withdrawn from production:

$$H_v = \frac{I_R\left(T_v^*\right)}{e^{\phi t}\bar{A}_{v-1}} = 1 - c\left(T_v^*\right). \tag{46}$$

5.2 The Contraction ($T_v^* \to T_v$)

During this phase, there is search, so that $H(t) > 0$. Since there is free entry into search, $w(t) = \delta\eta V^D(t)$, and so the value of entrepreneurship, $\delta\eta V^D(t)$, must grow at the same rate as the wage. Since the time until implementation for a successful entrepreneur is falling and there is no stream of profits (because implementation is delayed), the instantaneous interest rate necessarily equals ϕ:

$$r(t) = \frac{\dot{V}^D(t)}{V^D(t)} = \frac{\dot{w}(t)}{w(t)} = \phi. \tag{47}$$

Since $H(t) > 0$, $r(t) = \phi$ and $(1-\eta)V^D(t) > \Omega(t)$ it follows that the growth in the value of claims to unmatched ideas falls below the interest rate:

$$\frac{\dot{\Omega}(t)}{\Omega(t)} - \phi = -\frac{\delta H(t)}{S(t)}\left(\frac{(1-\eta)V^D(t) - \Omega(t)}{\Omega(t)}\right) < 0. \tag{48}$$

Since the accumulation of ideas is inherently irreversible ($\dot{z}(t) \geq 0$), it follows that during this phase, R&D optimally ceases and the expected value of an idea falls below the cost of producing it:

$$\frac{\mu\Omega(t)}{e^{\phi t}\bar{A}_{v-1}} < z(t) = z(T_v^*). \tag{49}$$

Intuitively, as the downturn proceeds, the likelihood that a given idea will be matched with a sector, before the subsequent boom, declines.

Lemma 3 *: During the downturn the value of an unmatched idea at time t is given by*

$$\Omega(t) = (1-\eta)V^D(t)\left(1 - \frac{S(T_v)}{S(t)}\right) + \frac{S(T_v)}{S(t)}e^{-\beta(t)}\Omega_0(T_v) \tag{50}$$

Since no R&D takes place during the downturn, the stock of potential knowledge grows at the rate ϕ until the beginning of the subsequent cycle, $Z_0(T_v) = e^{\phi[T_v - T_v^*]}Z(T_v^*)$. But since R&D is positive at the beginning of the next cycle, it must also be true that $\mu\Omega_0(T_v) = Z_0(T_v)$. Taken together these facts imply that while the value of claims to R&D falls during the downturn, their increase at the boom must exactly offset this: $e^{\phi[T_v - T_v^*]}\Omega(T_v^*) = \Omega_0(T_v)$. Combining this with (50) implies the following:

Proposition 1 *: The stock of unmatched ideas at the cyclical peak, $S(T_v^*)$, must satisfy*

$$\frac{(1-\eta)\mu e^{-\gamma}}{\eta\delta}\left(\frac{\Gamma_v}{\gamma S(T_v^*)(1 - e^{-\beta(T_v^*)}) + \Gamma_v e^{-\beta(T_v^*)}}\right) = e^{\gamma S(T_v^*)}. \tag{51}$$

Note that it must be the case that $\gamma S(T_v^*) > \Gamma_v$. Since the term in brackets must be less than unity, an additional necessary (but not sufficient) condition on parameters that must hold is $(1-\eta)\mu e^{-\gamma} > \eta\delta$. Equation (51) is analogous to (28) for the acyclical growth path — given growth in productivity at the boom, Γ_v, and the discount factor to the beginning of the next cycle, $\beta(T_v^*)$, it pins down the stock of unmatched ideas available at the previous cyclical peak.

Since $r(t) = \phi$, normalized consumption must decline during this phase:

$$\frac{\dot{c}(t)}{c(t)} = -\frac{\hat{\rho}}{\sigma} < 0. \tag{52}$$

This occurs during the downturn because labor gradually flows out of production and into commercialization.[16] Using (46) and (52), yields the following expression for aggregate effort allocated to commercialization at time t:

$$H(t) = 1 - e^{-\frac{\hat{\rho}}{\sigma}[t-T_v^*]}(1-H_v). \tag{53}$$

This, in turn, determines the measure of sectors in which commercially viable ideas are identified at each date:

$$-\dot{P}(t) = \delta H(t), \tag{54}$$

where $P(T_v^*) = 1$.[17] At the end of the cycle, the fraction of sectors that have identified commercially viable ideas is therefore

$$1 - P(T_v) = \int_{T_v^*}^{T_v} \delta H(\tau) d\tau. \tag{55}$$

5.3 The Implementation Boom

We denote the growth in aggregate productivity during the implementation period T_v by $\Gamma_v = \ln(\bar{A}_v/\bar{A}_{v-1})$. Since $\Gamma_v = \gamma\left(1 - P(T_v)\right)$, (55) and (53) determine the size of the boom as a function of the length of the downturn, $\Delta_v^C = T_v - T_v^*$:

$$\Gamma_v = \delta\gamma\Delta_v^C - \delta\gamma(1-H_v)\left(\frac{1-e^{-\frac{\hat{\rho}}{\sigma}\Delta_v^C}}{\hat{\rho}/\sigma}\right). \tag{56}$$

At the beginning of each cycle all labor is used in production. Since output is only augmented by the increase in aggregate productivity $C_0(T_v) = e^{\Gamma_v + \phi\Delta_v} C_0(T_{v-1})$. The Euler equation therefore implies a long run discount factor given by

$$R_0(T_v) - R_0(T_{v-1}) = \rho\Delta_v + \sigma(\Gamma_v + \phi\Delta_v) \tag{57}$$

[16] If ϕ is large enough, it is possible that the level of actual consumption, $C(t) = e^{\phi t}\bar{A}_{v-1}c(t)$, may continue to grow.

[17] The rate of change in P is given by $\frac{\dot{P}}{P} = -\delta h_i$. But since labor is allocated symmetrically to innovation only in the measure P of sectors where no innovation has occurred, $h_i = \frac{H}{P}$, so that $\dot{P} = -\delta H$.

During the expansion, (41) implies that the discount factor must grow by $\ln \frac{Z(T_v^*)}{Z(T_{v-1})} = -\phi\Delta^C + \ln \frac{Z(T_v)}{Z(T_{v-1})}$, recalling that there is no R&D in the recession. During the downturn the interest rate is ϕ. Combining these facts with (57), it follows that across the boom the discount factor must satisfy

$$\beta(T_v) = \rho\Delta_v + \sigma(\Gamma_v + \phi\Delta_v) - \ln \frac{Z(T_v)}{Z(T_{v-1})}. \tag{58}$$

Over the boom, the asset market must simultaneously ensure that entrepreneurs holding viable ideas are willing to implement immediately (and no earlier) and that, for households, holding equity in firms dominates holding claims to alternative assets (particularly stored intermediates). The following Proposition demonstrates that these conditions imply that during the boom, the discount factor must equal productivity growth:

Proposition 2 *: Asset market clearing at the boom requires that*

$$\beta(T_v) = \Gamma_v. \tag{59}$$

Since the interest rate is ϕ through the downturn it is also the case that $\beta\left(T_v^*\right) = \phi\Delta_v^C + \Gamma_v$ and that, using the household Euler equation:

$$\ln c_0(T_v) = \ln c(T_v^*) + \frac{\hat{\rho}\Delta_v^X - \Gamma_v}{\sigma}, \tag{60}$$

where $\Delta_v^X \equiv T_v^* - T_{v-1}$ denotes the expansion length.

5.4 Optimal Entrepreneurial Behavior

Why are entrepreneurs potentially willing to delay implementation to the boom? If they do so, they forego current flow profits until the boom, but they also do not reveal the content of their commercial improvement to other entrepreneurs.[18] This ensures that, at the forthcoming boom when they implement, they will be assured incumbency from that point. An additional advantage of this, apart from the boom being a time of enhanced economic activity, is that they will be ensured incumbency throughout the subsequent cycle, since any subsequent implementation will also be delayed.

The willingness of entrepreneurs to delay implementation until the boom and to just start engaging in search activities at exactly T_v^* depends crucially on the expected value of monopoly rents, relative to the current labor costs. This is a forward looking condition: given Γ and Δ^C, the present value of these rents depend crucially on the length of the subsequent cycle, $T_{v+1} - T_v$.

[18] We discuss evidence of this behavior in Section 9.

Since Lemma 2 implies that entrepreneurship starts at T_v^*, free entry into commercialization requires that

$$\delta\eta V^D(T_v^*) = \delta\eta e^{-\beta(T_v^*)} V_0^I(T_v) = e^{\phi\Delta_v^C} w_0(T_v). \tag{61}$$

Since the increase in the wage across cycles reflects only the improvement in productivity: $w_0(T_{v+1}) = e^{\Gamma_v + \phi\Delta_v} w_0(T_v)$, and since from the asset market clearing conditions, we know that $\beta(T_v^*) = \phi\Delta_v^C + \Gamma_v$, it immediately follows that the increase in the present value of monopoly profits from the beginning of one cycle to the next must, in equilibrium, reflect only the improvements in aggregate productivity:

$$V_0^I(T_{v+1}) = e^{\Gamma_v + \phi\Delta_v} V_0^I(T_v). \tag{62}$$

Equation (62) implies that, given some initial implementation period T_v, and values of Γ_v and Δ_v^C, the next implementation period, T_{v+1}, is determined. We therefore have the following result:

Proposition 3 *Given the boom size, Γ_v, the contraction length, Δ_v^C, and the dynamic path followed by $z(t)$, there exists a unique expansion length, Δ_v^X, such that entrepreneurs are just willing to commence search, Δ_v^C periods prior to the boom.*

It is worth contrasting the delay that entrepreneurs engage in here with that which would occur were we to simply assume that moving from successful R&D to commercial application takes time for exogenous reasons. Such an assumption would yield a lag between R&D and increases in productivity, however, there would be no reason for these increases in productivity to occur in the expansion, as they do in our model. Endogenous delay is crucial in generating pro-cyclical productivity because entrepreneurs delay implementation until the point in the cycle where demand conditions are most favorable. This is at the start of the expansion.

The equilibrium conditions on entrepreneurial behavior also impose the following requirements on our hypothesized cycle:

• Successful entrepreneurs at time $t = T_v$, must prefer to implement immediately, rather than delay implementation until later in the cycle or the beginning of the next cycle:

$$V_0^I(T_v) > V_0^D(T_v). \tag{63}$$

• Entrepreneurs who find commercially viable ideas during the downturn must prefer to wait until the beginning of the next cycle rather than implement earlier:

$$V^I(t) < V^D(t) \qquad \forall\, t \in (T_v^*, T_v) \tag{64}$$

• No entrepreneur wants to search for commercially viable ideas during the expansionary phase of the cycle. Since in this phase of the cycle $\delta V^D(t) < w(t)$, this condition requires that

$$\delta\eta V^I(t) < w(t) \qquad \forall\, t \in (0, T_v^*) \tag{65}$$

Finally, in constructing the equilibrium above, we have implicitly imposed the requirement that the downturn is not so long that commercially viable applications are identified in every sector:

$$P(T_v) > 0. \tag{66}$$

6 Stationary Cyclical Equilibrium Growth Path

We focus on a stationary cyclical equilibrium growth path along which the boom in productivity, Γ, and the length of each phase of the cycle (Δ^X, Δ^C) are constant. Along such a growth path, the potential productivity increases by

$$\ln \frac{Z(T_v)}{Z_0(T_{v-1})} = \phi\Delta + \Gamma = \phi\Delta + \gamma \left[N(T_v^*) - N(T_{v-1}) \right] \tag{67}$$

during the expansion, and an equal measure of ideas is matched with a profitable application during downturns. Combining (58), (59) and (67) yields the following implication:

Proposition 4 *: Along the stationary cyclical path, average long–run growth is given by*

$$\bar{g} = \phi + \frac{\hat{\rho}}{2-\sigma} = \frac{\rho + \phi}{2-\sigma} \tag{68}$$

Thus, long–run growth along this path is increasing in the rate of time preference and decreasing in the elasticity of intertemporal substitution. This is in sharp contrast to the acyclical growth path discussed in Section 3. To understand this, consider two key relationships. Firstly, the consumer's Euler equation implies, as usual, that a higher rate of return over the cycle yields a more rapidly growing consumption path. In equilibrium it follows that, on average,

$$\bar{g} = \frac{\bar{r} - \rho}{\sigma}, \tag{69}$$

where $\bar{r} = \left[R_0(T_v) - R_0(T_{v-1}) \right] / \Delta$ denotes the average interest rate over the cycle.

In the cyclical steady state, the rate of return must be sufficient to induce investment in R&D during the expansion, and to induce the financing of commercialization during the downturn. The former is given by the growth in the unit cost of R&D, $Z(t)/\mu$, during the expansion, $\Gamma + \phi\Delta^X$, and the latter equals the amount ensuring that implementation is delayed, $\Gamma + \phi\Delta^C$. It follows that the average rate of return over the entire cycle required to induce the investment that supports a growth rate $\bar{g}$ must be given by

$$\bar{r} = 2\frac{\Gamma}{\Delta} + \phi = 2\bar{g} - \phi. \tag{70}$$

Note that, in contrast to the acyclical steady–state, the required average rate of return is unambiguously increasing in the average growth rate.

Using (44), (56), (51), (60) and Proposition 2, the stationary cyclical equilibrium is fully described by the vector $(\Gamma, \Delta^X, \Delta^C, \hat{z}, \hat{S})$ and a recurring expansion path for consumption $\{\hat{c}(t)\}_{\tau=0}^{\Delta^X}$ which satisfy the following system:

$$\hat{c}(\tau) = F(\tau,\ e^{-\Gamma/\sigma + \frac{\hat{\rho}}{\sigma}\Delta^X}\hat{c}(\Delta^X),\ e^{-\Gamma+\phi\Delta}\hat{z}) \qquad \forall\ \tau \in [0, \Delta^X] \tag{71}$$

$$\Gamma = \delta\gamma\Delta^C - \delta\gamma\hat{c}(\Delta^X)\left(\frac{1 - e^{-\frac{\hat{\rho}}{\sigma}\Delta^C}}{\hat{\rho}/\sigma}\right) \tag{72}$$

$$\frac{(1-\eta)\mu e^{-\gamma}}{\eta\delta}\left(\frac{\Gamma}{\gamma\hat{S} + \left(\Gamma - \hat{S}\right)e^{-\Gamma-\phi\Delta^C}}\right) = e^{\gamma\hat{S}} = \hat{z} \tag{73}$$

$$\frac{e^{-\gamma}}{\delta\eta\gamma} = \frac{(1-e^{-\gamma})\,e^{-\Gamma}}{\gamma(1-e^{-\Gamma}) + \Gamma e^{-\Gamma}}\left[\int_0^{\Delta^X}\frac{e^{-\hat{\rho}\tau}}{e^{-\hat{\rho}\Delta^X}}\left(\frac{c(\tau)}{\hat{c}(\Delta^X)}\right)^{-\sigma}d\tau + \hat{c}(\Delta^X)\left(\frac{1-e^{-\frac{\hat{\rho}}{\sigma}\Delta^C}}{\hat{\rho}/\sigma}\right)\right] \tag{74}$$

$$(2-\sigma)\,\Gamma = \hat{\rho}(\Delta^X + \Delta^C) \tag{75}$$

Recall that the function $F(\cdot)$ represents the transitional dynamics during the expansion, given by (42) and (43). Although average growth depends only on preference parameters, technological parameters influence short–run growth and the nature of cycles.[19] In order to characterize these effects, however, we turn to numerical methods.

7 Numerical Analysis

7.1 Baseline Example

We numerically solve (71)–(75) for various combinations of parameters and check the existence conditions (63)–(66). We will first illustrate that the model can generate reasonable looking business cycles for parameter values that are within the regular bounds. We then explore the cyclicality of our key aggregates over these cycles. The parameters for our baseline example are given in Table 1.[20] The parameter γ implies an average labor share of about 0.7. We chose ρ and σ to yield a long run growth rate of 2% and an average risk–free real interest rate of 4% (these

[19] This dichotomy is not general. In the generalized model described in Section 8, technological parameters also affect long-run growth.

[20] The Gauss program used to generate the numerical simulations and the diagrams contained here is downloadable from the following URL: http://qed.econ.queensu.ca/pub/faculty/lloyd-ellis/research.html

values roughly correspond to average data for the post–war US.). In our baseline example, we assume no exogenous productivity growth, $\phi = 0$. We know of no way to measure the productivity parameters in R&D (μ), commercialization (δ) nor their respective shares (η) so we choose these to match a cycle length of approximately 8 years, with an expansion of almost 6 years and a contraction of just over 2 years. According to the NBER, the average business cycle in the US during the post-war period was about 6 years, but more recent cycles have tended to be longer, which is why we opted for this length. However, below we consider alternative combinations of parameters that yields both shorter and longer cycles.

Table 1: Baseline Parameters

Parameter	Value
δ	1.26
γ	0.30
σ	1.00
ρ	0.02
μ	0.67
η	0.20

Figure 4 depicts the evolution of some key aggregates over the cycle. The variable GDP is the sum of consumption, R&D expenditures and expenditures on commercialization. In the absence of productivity growth during the first phase, GDP is constant. However, it experiences an abrupt increase during the boom. During the second phase, GDP declines dramatically then continues a more gradual decline as the subsequent boom approaches. Although R&D expenditure peaks at the beginning of the cycle and then declines slowly during the first phase, it is still strongly correlated with the major movements in consumption and GDP. Even if we include the wage costs associated with search in an aggregate measure of "all innovation" expenditure, this aggregate remains strongly pro-cyclical. If we de-trend the logs of GDP and aggregate innovation expenditure, the correlation between them is 0.88. Note that, as illustrated in Figure 5, the stock of potentially productive knowledge grows steadily during expansions and comes to a halt during contractions.

Figure 5 illustrates the factors affecting the incentives to search for commercially viable ideas and to implement them at each stage of the cycle. During expansions, wages are relatively high and the subsequent boom is far away. Consequently, the values of newly commercialized ideas lie below the unit costs of search effort, whether or not implementation is immediate or delayed. Eventually, as the next boom approaches, the value of delayed implementation becomes high enough to warrant the cost of search effort, and commercialization starts to pick up. The

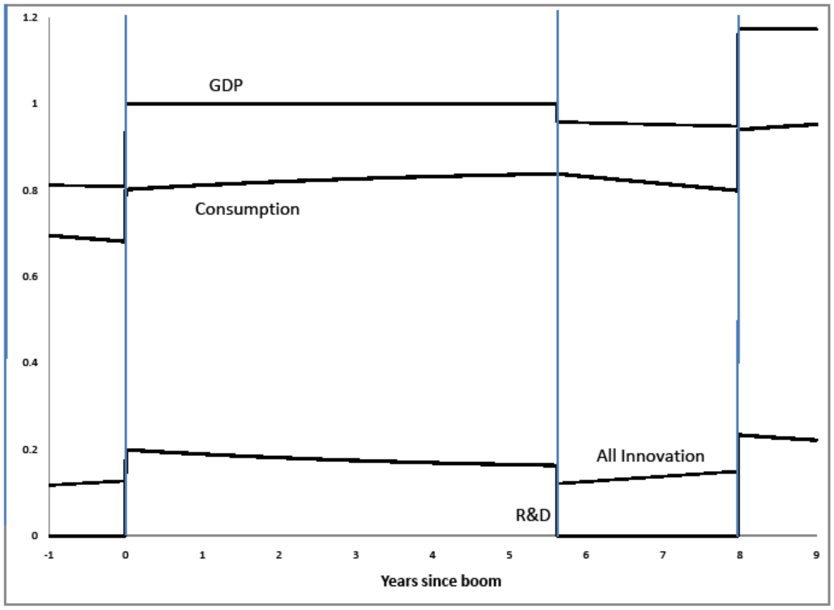


Figure 4: Evolution of Key Aggregates in Baseline Example

value of immediate implementation remains below that of delay because of the risk to profits of implementing too early.

Figure 6 illustrates the factors affecting the incentives to undertake R&D at each stage of the cycle. The "value of a new idea" corresponds to $\Omega(t)$ in the model and "implementation probability" refers to the probability that a commercially viable application will be found for an existing idea prior to the subsequent boom. Since commercialization is concentrated during the contraction, the implementation probability is constant during the expansion. However, as the contraction proceeds, the likelihood that any new idea created by the R&D sector will find a commercially viable application before the next boom falls gradually to zero. Thus, the value of new ideas grows with the unit cost of generating them (proportional to the knowledge stock) during the expansion, but then falls with the declining implementation probability during the contraction.

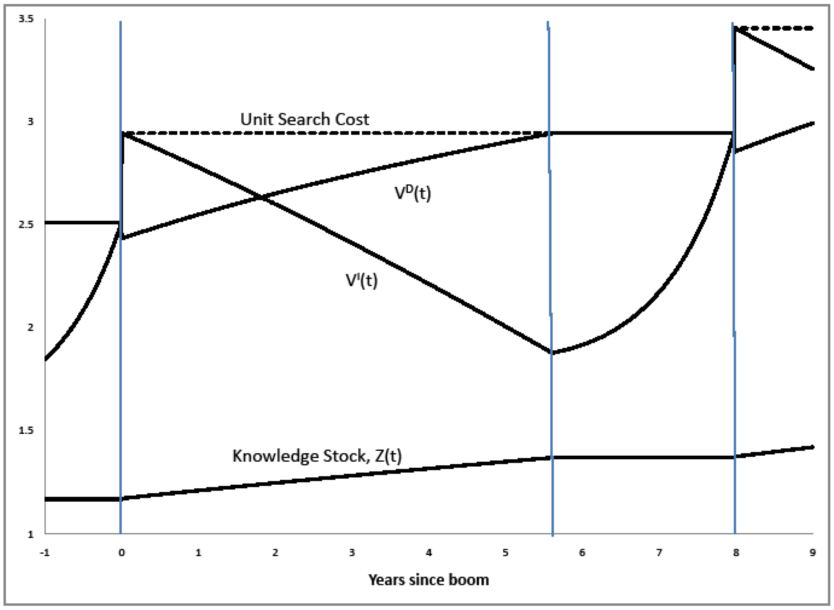


Figure 5: Incentives to search for commercially-viable ideas

Table 2: Comparative Stationary Cycles

Parameters		Γ	Δ^X	Δ^C	$\hat{z}$	$\bar{g}$ (%)	$\hat{S}$	$P(T)$	Corr(y, I)
Baseline		0.16	5.62	2.36	1.37	2.00	1.06	0.47	0.88
$\delta =$	1.25	0.19	6.81	2.80	1.40	2.00	1.12	0.36	0.84
	1.27	0.11	3.99	1.72	1.34	2.00	0.97	0.62	0.94
$\gamma =$	0.299	0.19	6.77	2.78	1.40	2.00	1.12	0.36	0.84
	0.301	0.13	4.54	1.95	1.35	2.00	1.00	0.57	0.92
$\sigma =$	0.99	0.11	3.99	1.72	1.34	1.98	0.98	0.62	0.94
	1.01	0.21	7.38	3.02	1.41	2.02	1.14	0.30	0.81
$\rho =$	0.0199	0.14	4.80	2.04	1.36	1.99	1.02	0.55	0.91
	0.0201	0.18	6.48	2.68	1.39	2.01	1.10	0.39	0.85
$\mu =$	0.15	0.15	5.48	2.03	1.17	2.00	0.53	0.50	0.93
	0.25	0.17	5.77	2.77	1.57	2.00	1.50	0.43	0.82
$\eta =$	0.199	0.18	6.49	2.67	1.40	2.00	1.11	0.39	0.85
	0.201	0.14	4.81	2.06	1.35	2.00	1.01	0.54	0.91

Table 2 details the consequences of varying each of the parameters of the model around the baseline example. The nature of the cycle is quite sensitive to parameter changes and the size of the changes considered is partly dictated by the desire to generate cycle lengths in the 2-10 year range. As noted earlier, changes in technological parameters have no impact on long run growth,

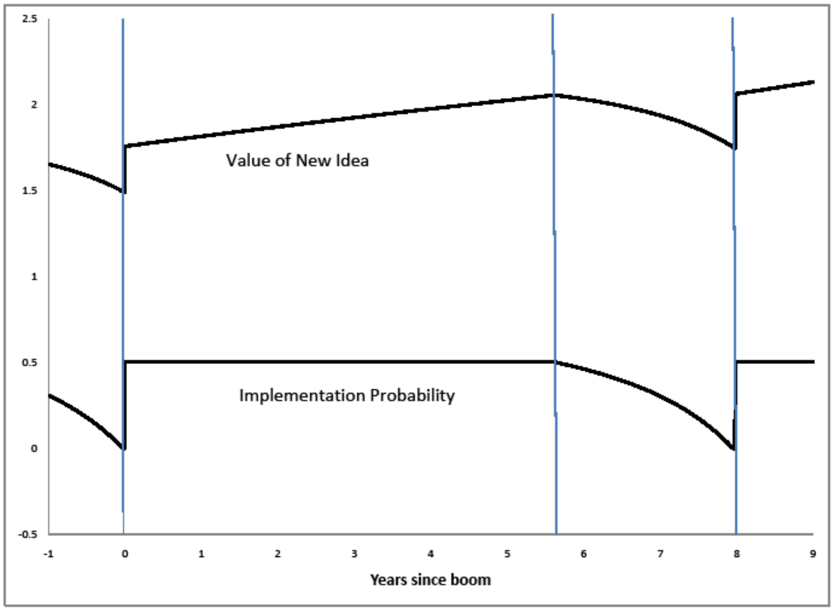


Figure 6: Incentives to undertake R&D

but do affect growth in the short run and the lengths of each phase of the cycle. Note that, as indicated by the column headed $P(T)$, the fraction of sectors that experience no improvements is often large. Nevertheless every sector experiences a boom in demand as a result of spillovers from others' success; i.e., the aggregate upturn.

Increases in the commercialization success rate, δ, or the size of productivity increments, γ, shorten the length of both phases of the cycle. The length of the contraction declines because a higher rate of success induces entrepreneurs to want to implement earlier. Consequently, the size of the productivity boom declines, inducing less R&D and a shorter expansion. Overall, these adjustments are such that the steady state average growth rate remains unchanged. In contrast, an increase in the productivity of R&D, μ, lengthens both phases of the cycle and increases the size of the boom.

The last column in Table 2 gives the correlation between the de-trended logs of GDP and total expenditures on innovation implied by the model. As can be seen, these variables are strongly (though not perfectly) correlated.[21] Any parameter change which increases the length of the cycle tends to reduce this correlation, because the major shifts at the beginning of each phase become

[21] The correlation between GDP and R&D only is even stronger.

less dominant determinants. However, for all parameter combinations that we have considered, the correlation is above 0.75.

7.2 Positive Exogenous Growth ($\phi = 0.01$)

Allowing for a small amount of positive exogenous growth, $\phi > 0$, does not change the qualitative features of the cycle, except that GDP grows at a positive rate during the expansion. Figure 7 illustrates the cyclical dynamics of consumption, GDP and innovation when we allow for some positive exogenous growth. Specifically, relative to the baseline case, we set $\phi = 0.01$ and reduce ρ to 0.01, so that long run growth is still $\bar{g} = 0.02$. Because $\rho = \phi$ and $\sigma = 1$, consumption no longer declines during the second phase of the cycle, but is exactly constant. Indeed, if we were to set parameters so that $\rho < \phi$, consumption would continue to grow. Properly measured aggregate GDP grows steadily through the expansion, drops at the beginning of the second phase, but then grows at a relatively low rate until the boom. Thus, in this example, the "recession" in GDP is actually very short, but is followed by a period of sluggish growth.

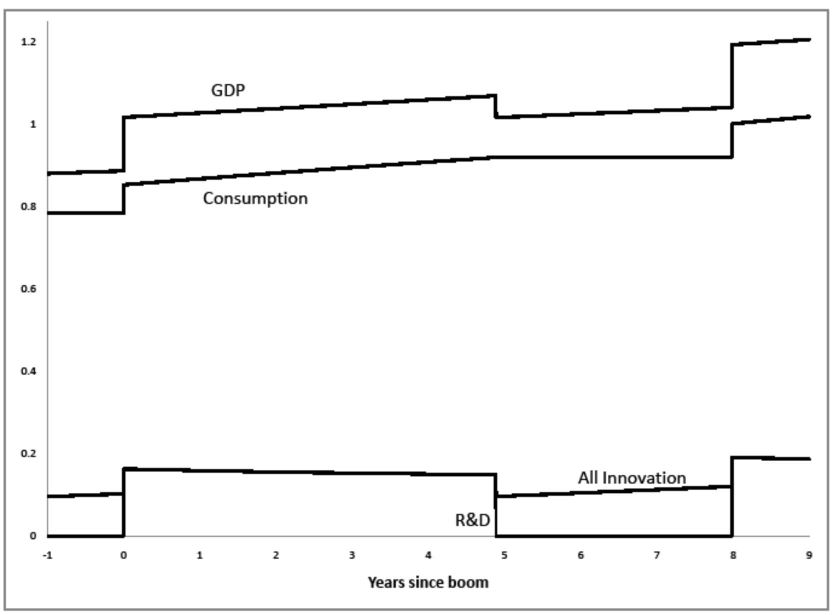


Figure 7: Evolution of aggregates with $\phi = 0.01$

8 Generalization: Production Labor

In the basic model, we have only considered the role of a small part of the overall labour force in the form of entrepreneur/managers. We now allow for a second type of production labor that can only be used in the production of consumption goods and R&D.[22] We assume that each household inelastically supplies $\bar{L} > 1$ units of production labour. Following Hansen (1985), we now assume that expected household utility is given by

$$U(t) = \int_t^\infty e^{-\rho(\tau - t)} \left(\ln C(\tau) - b\bar{L}\lambda(\tau)\right) d\tau \tag{76}$$

where $b > 0$ and $\lambda(t)$ denotes the probability that the production labor supplied by each household is employed at date t. We continue to assume that each household inelastically supplies 1 unit of managerial/entrepreneurial human capital.[23] Each household maximizes (1) subject to the intertemporal budget constraint

$$\int_t^\infty e^{-[R(\tau)-R(t)]} C(\tau) d\tau \leq B(t) + \int_t^\infty e^{-[R(\tau)-R(t)]} \left[w(\tau) + w_p(\tau)\lambda(\tau)\bar{L}\right] d\tau, \tag{77}$$

where $w_p(t)$ denotes the production wage.

Final output is now produced according to a Cobb-Douglas production function utilizing intermediate goods and services and production labour, $L(t)$:

$$Y(t) = X(t)^\alpha L(t)^{1-\alpha}, \tag{78}$$

where

$$X(t) = \exp\left(\frac{\phi}{\alpha} t + \int_0^1 \ln x_i(t) di\right).$$

Output of intermediate i depends upon the state of technology in sector i, $A_i(t)$, and the the human capital of entrepreneur/managers l_i, according to:

$$x_i^s(t) = A_i^{\frac{1}{\alpha}}(t) l_i(t). \tag{79}$$

In addition to the equilibrium conditions described in Section 2.3, these changes also imply that a first-order condition for production-labor supply must be respected:

$$w_p(t) = bC(t) \tag{80}$$

Factor incomes are now given by $\pi(t) = \alpha(1 - e^{-\gamma})Y(t)$, $w(t)(1 - H(t)) = \alpha e^{-\gamma} Y(t)$ and

$$w_p(t)\lambda(t)\bar{L} = (1 - \alpha)Y(t) \tag{81}$$

[22] We do not think of this type of labor as comprising only unskilled workers. Rather it includes those workers in a non-supervisory or entrepreneurial role.

[23] We could include the disutility associated with this type of labor in household preferences. However, since in equilibrium they are always employed, this would amount to subtracting a constant term.

Finally, market clearing in the market for production labor implies that

$$\lambda(t)\bar{L} = L(t). \tag{82}$$

These changes make very little difference to the qualitative nature of the cyclical equilibrium growth path.[24] However, there are now additional implications for cyclical movements in wages and employment.[25] Relative to the baseline example we set $\alpha = 0.8$, $\gamma = 0.5$ and $b\bar{L} = 0.05$. This combination of parameters implies an overall labor share in production of $1 - \alpha(1 - e^{-\gamma}) = 68\%$ and a ratio of production labor to managerial human capital of 4:1. We then set $\delta = 0.46$ and $\phi = 0.005$ to yield the same 8 year cycle length as in the baseline example with 2% average growth. Figure 8 illustrates the implications for the production wage and employment. As can be seen, both are pro-cyclical, but employment is much more so, falling precipitously during the downturn. This is because employment tracks output fairly closely, whereas the production wage tracks consumption.

Figure 8: Employment and wages in the generalized example

One other qualitative difference from the basic version of the model is that long run growth is no longer independent of technological parameters. The greater is the share of production labor,

[24] The acyclical growth path is identical to that without production labor.
[25] Mathematical details are available from the authors upon request.

$1-\alpha$, the more sensitive is long–run growth to these parameters. In particular, higher values of δ and γ, which determine the productivity of commercialization, imply higher growth rates.

9 Discussion

In this section we discuss the extent to which the aggregate patterns identified in the model match those in the data. While the empirical evidence is somewhat disparate, we argue that it is consistent with the key mechanisms that drive the cycle displayed by our model.

9.1 Delays between discovery, patenting and implementation

Splitting up innovation into two phases allows for a stochastic lag between the timing of initial discoveries and the dates when they are put to practical use. Barlevy (2007) argues that while such a decomposition may be realistic, it is unlikely to overturn the inherent counter-cyclicality of R&D in Schumpeterian models. Here we have shown that, in fact, these two components of innovation can exhibit very different cyclical properties in equilibrium. A key source of these cyclical properties is the fact that firms optimally delay implementation. However, while such delays are theoretically possible, an important question is: how important is such behavior in reality?

The fact that firms often apply for patents soon after undertaking research seems to go against the delay hypothesis. However, the timing of a patent application does not, in general, coincide with implementation. As Lemley and Shapiro (2005) point out, many issued patents turn out to have very little commercial significance. For those that do, it is often the case that the issuance of a patent can be effectively delayed for many years until a commercial application is found. Graham (2004, p.1) discusses the famous example of George Selden's "Road Engine". Soon after the patent was finally issued in 1895, Selden was commanding royalties of 1.25% on the retail value of every automobile sold in the United States. However, Selden's original patent application was filed in 1879, some 16 years before the actual patent was granted. Despite this, he was able to claim royalties on cars, whose inception and mass production was not to take place for almost twenty years.

Graham (2004) studies the strategy of delaying disclosure through the use of "continuation patents" which effectively protect the applicant's right to an idea and a share of the profits that subsequently accrue when related technologies find commercial applications. He finds that these sorts of delays are not historically isolated examples: continuation patents comprised some 20% of patents issued in the period 1975-2001 in the US. By following such a strategy, patent applicants are able to keep details of patents secret while being approved by the patent board – a mean delay of around two years – and, in "continuation" cases, for an average of two more years beyond that. This suggests that, for these cases at least, the operative delay to communication of an idea tends to be its implementation in production.

9.2 Pro-cyclicality of R&D, TFP and Implementation

As discussed in the introduction, most empirical evidence finds that both the inputs and outputs of R&D are pro-cyclical. The fact that productivity is pro-cyclical is consistent with the idea that implementation of productivity improvements is also strongly pro-cyclical. However, more direct evidence in support of this interpretation is provided by Geroski and Walters (1995) who investigate the cyclical properties of the implementation of major innovations in the UK during the post-war period. They find that the implementation of innovations is pro-cyclical and that they occur in small clusters. These observations are broadly consistent with the pattern displayed in our model.

We have assumed that none of the counter-cyclical "commercialization" is picked up in observed measures of R&D. However, although commercialization is unlikely to be fully picked up, it is likely that some of this finds its way into reported R&D. To address this in the model, we consider the extreme case of full attribution of commercialization to R&D. The result of this is indicated in Figure 4. The finer "All Innovation" line, which overlaps with the solid R&D investment line in expansions measures total expenditures on innovation over the cycle. As can be readily seen, this is still quite pro-cyclical, and remains so for all reasonable parameterizations of the model that we have experimented with.

Although TFP in our model is pro-cyclical (because its correlation with GDP is dominated by the effects of the boom), it does not decline during downturns. However, during this second phase of our cycle, human capital is re-allocated out of production. This is not generally going to be properly measured in the data, since a lot of what we call commercialization is likely to be a reallocation within firms (or even within people). Consequently, TFP, measured using a conventional value added production function, would appear to fall during the downturn. For the same reasons, labor share would also appear to decline during the downturn, a stylized fact that has recently been emphasized by Rios-Rull and Santaeulalia-Llopis (2006).

9.3 Counter-cyclicality of Commercialization

Many of the activities that constitute the search for commercially-viable ideas are not measured as part of R&D. These activities are likely to be undertaken by entrepreneurs, or performed in-house by firms' management. In order to assess the cyclical behavior of these activities, the usual aggregate data sets are not helpful since much of it occurs without separately measured expenditures or occupational reallocation. A number of studies have used either specialized data sets based on surveys or proxies to obtain related estimates.

Perhaps the most direct evidence is provided by Nickell, Nicolitsas and Patterson (2001). They investigate the cyclical behavior of managerial innovations occuring in downturns using 2 unique data sets. The first, based on the Confederation of British Industries Pay Databank includes information on two measures of innovation: the removal of restrictive practices and the introduction of new technology. The second surveys small to medium sized manufacturing firms in engineering, plastics, electronics and food, drink and tobacco industries and provides information on managerial innovations such as changes in structure, more decentralization, changes in human resources management practices and the introduction of just in time technologies. Both data sets support the view that when demand is slack and profitability low, managers and workers devote more time to innovative activities.

Some circumstantial evidence comes from "labor hoarding": the employment of labor during recessions beyond that which is technologically necessary to meet production requirements. This is commonly modelled as arising from labour adjustment costs due to hiring and firing. However, an alternative explanation is that part of this excess labor is being used to adapt ideas and approaches that will be useful in the future. In their classic survey of US manufacturing plants, Fay and Medoff (1985) found that, during a trough quarter, the typical plant paid for about 8 percent more labor hours than technologically necessary, and about half of which was used in productive activities. Of the 50% of respondents that re-assigned workers during recessions, about one third allocated them to "reworking output".

More evidence comes from the type of workers employed over the cycle. Innovative activities are more likely to require skilled, non–production workers, so that during downturns the ratio of skilled to unskilled workers should rise. Although this is typically the case in the data, it is possible that this can be explained by the fact that costs of adjustment for skilled workers are relatively high. However, such a motivation would not lead to an absolute increase in skilled employment during downturns. Using Spanish Manufacturing data, Aguirregabiria and Alonso-Borrego (2001) actually find the level of employment of skilled workers to be significantly countercyclical.

A final piece of evidence that is consistent with the counter-cyclicality of commercialization, is the behavior of the stock market during recession. As Francois and Lloyd-Ellis (2008) document, during all but one post–war US recession, while investment fell the value of Tobin's Q quickly reached a trough early and then rose continuously throughout the recession, well ahead of the subsequent increases in investment (on average 4 quarters). If movements in Tobin's Q largely reflect increases in the value of intangibles, this behavior is consistent with an increase in the stock of commercially-viable ideas during downturns.

10 Concluding Remarks

The theory of cycles that we have developed here is "Schumpeterian" in the sense that downturns induce innovative activities, in the form of commercialization, which play a positive role in promoting long term growth. Nevertheless R&D investment (and the associated labor allocation) in our model is clearly pro-cyclical. This results from three assumptions we have made about R&D that distinguish it from other forms of innovative activity: (1) it is enhanced by implemented knowledge, (2) it is a long-term investment with uncertain applications and (3) it suffers from diminishing returns over time.

Barlevy (2007) conjectures that in a model where both the creation of basic ideas, and their development, were allowed as two separate phases it would continue to be the case that both would tend to be biased towards booms, as the development stage is in his model. The intuitive reasoning is that, as long as shocks are persistent, the occurrence of a recession today should increase the probability of lower profits for the implementation a number of periods from now. The reason this does not occur here is because shocks are not driven by an exogenously persistent process, they are part of an anticipated cycle in activity. Forward-looking entrepreneurs know that although the economy is heading into recession, the recession is of finite length and its end can be (with some error) anticipated. The matching of basic ideas with commercial applications thus occurs with an eye to this future upturn and occurs in recessions. It is worth noting, however, that even if all of this recessionary innovative activity is measured as part of R&D, which it is unlikely to be, overall R&D is still pro-cyclical.

Some features of our model's prediction are clearly at odds with the facts. However, it is possible to extend the model in various ways to address some of these issues. In particular, the productivity boom and the associated jump in output are rather abrupt. As we show in a recent paper, Francois and Lloyd–Ellis (2008), adding capital can help to smooth out the boom to some extent. Alternatively, allowing for the implementation process to be stochastic, so that not all ideas are implemented simultaneously, yields a less abrupt upturn. A further unrealistic feature of the cyclical process that we generate is that every cycle is the same, and all fluctuations are deterministic. Extending the model to allow for some stochastic elements relaxes some of these strong predictions. In particular, temporary i.i.d. shocks can change the length and amplitude of each cycle without changing the basic story. Here we have abstracted from these features in order to focus on the cyclical behavior of innovative activities. However, it seems likely that adding all of these features into one model – though cumbersome – would not change the main results, nor this paper's main conclusion, which is that Schumpeterian theories of the business cycle can be perfectly consistent with pro–cyclical R&D.

Appendix

Proof of Lemma 1 We show: (1) that if a signal of success from a potential entrepreneur is credible, other entrepreneurs stop innovation in that sector; (2) given (1) entrepreneurs have no incentive to falsely claim success.

Part (1): If entrepreneur i's signal of success is credible then all other entrepreneurs believe that i has a productivity advantage which is e^{γ} times better than the existing incumbent. If continuing to innovate in that sector, another entrepreneur will, with positive probability, also develop a productive advantage of e^{γ}. Such an innovation yields expected profit of 0, since, in developing their improvement, they do not observe the non-implemented improvements of others, so that both firms Bertrand compete with the same technology. Returns to attempting innovation in another sector where there has been no signal of success, or from simply working in production, $w(t) > 0$, are thus strictly higher, .

Part (2): If success signals are credible, entrepreneurs know that upon success, further innovation in their sector will cease from Part (1) by their sending of a costless signal. They are thus indifferent between falsely signalling success when it has not arrived, and sending no signal. Thus, there exists a signalling equilibrium in which only successful entrepreneurs send a signal of success.

Proof that w, is pinned down by the level of technology: From (14) $x_i^d(t) = \frac{Y(t)}{p_i(t)}$, so that from (15) $Y(t) = x_i^d(t)\frac{w(t)}{e^{-\gamma}A_i(t)}$, but since the intermediate technology is linear (from (5) this is $x_i^s(t) = A_i(t)l_i(t)$ and $x_i^s(t) = x_i^d(t)$ in equilibrium) we thus have $Y(t) = l_i(t)\frac{w(t)}{e^{-\gamma}}$. Substituting from the final output production function (3) and substituting again for x_i yields: $\exp\left(\phi t + \int_0^1 \ln\left[A_i(t)l_i(t)\right] di\right) = l_i(t)\frac{w(t)}{e^{-\gamma}}$. But the symmetry of sectors implies, again in equilibrium, that $l_i(t) = l(t)\,\forall i$, so that we have $\exp\left(\phi t + \int_0^1 \ln\left[A_i(t)\right] di\right) = \frac{w(t)}{e^{-\gamma}}$. Rearranging yields: $w(t) = \exp\left(\phi t + \int_0^1 \ln\left[A_i(t)\right] di\right) e^{-\gamma} \equiv e^{\phi t}\overline{A}_{v-1}e^{-\gamma}$.■

Proof of Lemma 2: There are two possible alternatives which can be ruled out by contradiction. (1) Suppose instead that at T_v^*, $\dot{Z} = 0$ and $H = 0$. From (33) it follows that $\dot{\Omega}/\Omega = r(T_v^*) > \phi$. But then $\mu\Omega(T_v^*) > Z(T_v^*)$, so there would be entry into R&D, contradicting the supposition. (2) Suppose instead that at T_v^*, $\dot{Z} > 0$ and $H > 0$. Free entry into search implies that $r(t) = \dot{V}^D/V^D = \phi$. It follows from (33) that $\dot{\Omega}/\Omega - \phi < 0$. But then $\mu\Omega(T_v^*) < z(T_v^*)$, so there would be no entry into R&D, contradicting the second supposition.■

Proof of Lemma 3: During the downturn the value of untapped ideas can be expressed as

$$
\begin{aligned}
\Omega(t) &= (1-\eta)\int_t^{T_v} e^{-\int_t^\tau \frac{\delta H(s)}{S(s)}ds}\frac{\delta H(\tau)}{S(\tau)}V^D(\tau)d\tau + e^{-\beta(t)-\int_t^{T_v}\frac{\delta H(\tau)}{S(\tau)}d\tau}\Omega_0(T_v) \\
\Omega(t) &= -(1-\eta)V^D(t)\int_t^{T_v} e^{\int_t^\tau \frac{\dot{S}(s)}{S(s)}ds}\frac{\dot{S}(\tau)}{S(\tau)}d\tau + e^{-\beta(t)+\int_t^{T_v}\frac{\dot{S}(\tau)}{S(\tau)}d\tau}\Omega_0(T_v) \\
\Omega(t) &= -(1-\eta)V^D(t)\int_t^{T_v} e^{\ln S(\tau)-\ln S(t)}\frac{\dot{S}(\tau)}{S(\tau)}d\tau + e^{-\beta(t)+\ln S(T_v)-\ln S(t)}\Omega_0(T_v) \\
\Omega(t) &= -(1-\eta)\frac{V^D(t)}{S(t)}\int_t^{T_v}\dot{S}(\tau)d\tau + e^{-\beta(t)+\ln S(T_v)-\ln S(t)}\Omega_0(T_v) \\
\Omega(t) &= (1-\eta)V^D(t)\left(1-\frac{S(T_v)}{S(t)}\right)+\frac{S(T_v)}{S(t)}e^{-\beta(t)}\Omega_0(T_v)
\end{aligned}
$$

Proof of Proposition 1: Since no new ideas emanate form the R&D sector between dates T_v^* and T_v, it follows that $e^{\phi[T_v-T_v^*]}\Omega(T_v^*)=\Omega_0(T_v)$. Using (50) we can write

$$
\begin{aligned}
\Omega(T_v^*)\left(1-\frac{S(T_v)}{S(T_v^*)}e^{-\beta(T_v^*)}\right) &= (1-\eta)V^D(T_v^*)\left(\frac{S(T_v^*)-S(T_v)}{S(T_v^*)}\right) \\
\Omega(T_v^*) &= (1-\eta)V^D(T_v^*)\left(\frac{S(T_v^*)-S(T_v)}{S(T_v^*)-S(T_v)e^{-\beta(T_v^*)}}\right) \\
\Omega(T_v^*) &= \frac{(1-\eta)e^{-\gamma}e^{\phi T_v^*}\bar{A}_{v-1}}{\eta\delta}\left(\frac{S(T_v^*)-S(T_v)}{S(T_v^*)-S(T_v)e^{-\beta(T_v^*)}}\right)
\end{aligned}
$$

We know that $\mu\Omega(T_v^*)=Z(T_v^*)=e^{-\phi[T_v-T_v^*]}Z(T_v)=e^{-\phi[T_v-T_v^*]}e^{\phi T_v}\bar{A}_v e^{\gamma S(T_v)}$, and so

$$
\begin{aligned}
e^{-\phi[T_v-T_v^*]}e^{\phi T_v}\bar{A}_v e^{\gamma S(T_v)} &= \frac{(1-\eta)\mu e^{-\gamma}e^{\phi T_v^*}\bar{A}_{v-1}}{\eta\delta}\left(\frac{S(T_v^*)-S(T_v)}{S(T_v^*)-S(T_v)e^{-\beta(T_v^*)}}\right) \\
e^{\Gamma_v+\gamma S(T_v)} &= \frac{(1-\eta)\mu e^{-\gamma}}{\eta\delta}\left(\frac{S(T_v^*)-S(T_v)}{S(T_v^*)-S(T_v)e^{-\beta(T_v^*)}}\right)
\end{aligned}
$$

But $S(T_v)=S(T_v^*)-\Gamma_v/\gamma$, so that

$$
e^{\gamma S(T_v^*)}=\frac{(1-\eta)\mu e^{-\gamma}}{\eta\delta}\left(\frac{\Gamma_v}{\gamma S(T_v^*)-\left(\gamma S(T_v^*)-\Gamma_v\right)e^{-\beta(T_v^*)}}\right)
$$

Re-arranging yields (51).

Proof of Proposition 2: For an entrepreneur who is holding a commercial viable idea, $\eta V^I(t)$ is the value of implementing immediately. Just prior to the boom, when the probability of displacement is negligible, the value of implementing immediately must equal that of delaying until the boom:

$$
\delta\eta V^I(T_v)=\delta\eta V^D(T_v)=w(T_v). \tag{83}
$$

During the boom, since entrepreneurs prefer to implement immediately, it must be the case that $V_0^I(T_v)>V_0^D(T_v)$. Thus the return to innovation at the boom is the value of immediate (rather than delayed) incumbency. It follows that free entry into entrepreneurship at the boom requires that

$$\delta\eta V_0^I(T_v) \leq w_0(T_v) \tag{84}$$

The opportunity cost to financing entrepreneurship is the rate of return on shares in incumbent firms in sectors where no innovation has occurred. Just prior to the boom, this is given by the capital gains in sectors where no innovations have occurred

$$\beta(T_v) = \log\left(\frac{V_0^I(T_v)}{V^I(T_v)}\right). \tag{85}$$

Note that since the short–term interest rate is zero over this phase, $\beta(t) = \beta(T_v)$, $\forall\ t \in (T_v^*, T_v)$. Combined with (83) and (84) it follows that asset market clearing at the boom requires

$$\beta(T_v) \leq \log\left(\frac{w_0(T_v)}{w(T_v)}\right) = \Gamma_v. \tag{86}$$

If innovative activities are to be financed at time t, households cannot be strictly better off buying claims to stored intermediate goods. In sectors with unimplemented innovations, entrepreneurs who hold innovations have the option of implementing immediately but not actually selling until the boom. The benefit of doing this comes from lower relative cost of labor prior to the boom. It follows that the long run rate of return on claims to firm profits just prior to the boom must satisfy

$$\beta(T_v) \geq \log\left(\frac{w_0(T_v)}{w(T_v)}\right) = \Gamma_v. \tag{87}$$

Combining (86) and (87) yields the result.

Proof of Proposition 3: The discounted monopoly profits from owning an innovation at time T_{v-1} is given by

$$\begin{aligned}
V_0^I(T_{v-1}) &= (1-e^{-\gamma})\int_{T_{v-1}}^{T_v} e^{-\int_{T_{v-1}}^{\tau} r(s)ds}Y(\tau)d\tau + P(T_v)e^{-\beta(T_{v-1})}V_0^I(T_v) \\
&= (1-e^{-\gamma})\bar{A}_{v-1}e^{\phi T_{v-1}}\Bigg[\int_{T_{v-1}}^{T_v^*} e^{-\int_{T_{v-1}}^{\tau}[r(s)-g(s)]ds}d\tau \\
&\quad + e^{-\int_{T_{v-1}}^{T_v^*} r(s)ds}e^{\phi(T_v^*-T_{v-1})}(1-H_v)\int_{T_v^*}^{T_v} e^{-\phi(\tau-T_v^*)}e^{-\frac{\rho-\phi}{\sigma}(\tau-T_v^*)}d\tau\Bigg] + P(T_v)e^{-\beta(T_{v-1})}V_0^I(T_v) \\
&= (1-e^{-\gamma})\bar{A}_{v-1}e^{\phi T_{v-1}}\left[\int_{T_{v-1}}^{T_v^*} e^{-\int_{T_{v-1}}^{\tau}[r(s)-g(s)]ds}d\tau + e^{-\Gamma_v}(1-H_v)\int_{T_v^*}^{T_v} e^{-\frac{\hat{\rho}}{\sigma}(\tau-T_v^*)}d\tau\right] \\
&\quad +P(T_v)e^{-\beta(T_{v-1})}V_0^I(T_v) \\
&= (1-e^{-\gamma})\bar{A}_{v-1}e^{\phi T_{v-1}}\left[\int_{T_{v-1}}^{T_v^*} e^{-\hat{\rho}(\tau-T_{v-1})}\left(\frac{c(\tau)}{c_0(T_{v-1})}\right)^{-\sigma}d\tau + e^{-\Gamma_v}(1-H_v)\left(\frac{1-e^{-\frac{\hat{\rho}}{\sigma}\Delta_v^E}}{\hat{\rho}/\sigma}\right)\right] \\
&\quad +P(T_v)e^{-\beta(T_{v-1})}V_0^I(T_v)
\end{aligned}$$

Substituting for $V_0^I(T_v)$ using (62), and rearranging yields

$$V_0^I(T_{v-1}) = \left(\frac{(1-e^{-\gamma})\bar{A}_{v-1}e^{\phi T_{v-1}}}{1-P(T_v)e^{\Gamma_v+\phi\Delta_v-\beta(T_{v-1})}}\right)\left[\int_{T_{v-1}}^{T_v^*} e^{-\hat{\rho}(\tau-T_{v-1})}\left(\frac{c(\tau)}{c_0(T_{v-1})}\right)^{-\sigma} d\tau + e^{-\Gamma_v}(1-H_v)\left(\frac{1-e^{-\frac{\hat{\rho}}{\sigma}\Delta_v^C}}{\hat{\rho}/\sigma}\right)\right]. \tag{88}$$

Noting that $\delta\eta V_0^I(T_{v-1}) = w_0(T_{v-1}) = e^{-\gamma}\bar{A}_{v-1}e^{\phi T_{v-1}}$, that $P(T_v) = 1-\Gamma_v/\gamma$ and that $\beta(T_{v-1}) = \sigma(\Gamma_v+\phi\Delta_v)+\rho\Delta_v$, we can express this as

$$\frac{e^{-\gamma}}{\delta\eta} = \left(\frac{(1-e^{-\gamma})}{1-(1-\Gamma_v/\gamma)e^{(1-\sigma)\Gamma_v-\hat{\rho}\Delta_v}}\right)\left[\int_{T_{v-1}}^{T_v^*} e^{-\hat{\rho}(\tau-T_{v-1})}\left(\frac{c(\tau)}{c_0(T_{v-1})}\right)^{-\sigma} d\tau + e^{-\Gamma_v}(1-H_v)\left(\frac{1-e^{-\frac{\hat{\rho}}{\sigma}\Delta_v^C}}{\hat{\rho}/\sigma}\right)\right]$$

Using the fact that $c_0(T_{v-1}) = e^{-\Gamma/\sigma+\frac{\hat{\rho}}{\sigma}\Delta^X}c(T_v^*)$, we can write this, after some rearrangement, as

$$\frac{e^{-\gamma}}{\gamma\delta\eta} = \left(\frac{(1-e^{-\gamma})e^{-\Gamma_v}}{\gamma(1-e^{-\Gamma_v})+\Gamma_v e^{-\Gamma_v}}\right)\left[\int_{T_{v-1}}^{T_v^*} \frac{e^{-\hat{\rho}(\tau-T_{v-1})}}{e^{-\hat{\rho}\Delta_v^X}}\left(\frac{c(\tau)}{c(T_v^*)}\right)^{-\sigma} d\tau + (1-H_v)\left(\frac{1-e^{-\frac{\hat{\rho}}{\sigma}\Delta_v^C}}{\hat{\rho}/\sigma}\right)\right]$$

Since $c(T_v^*) = 1-H_v$ and since $\Delta_v^X = T_v^* - T_{v-1}$, this can be expressed as

$$\frac{e^{-\gamma}}{\gamma\delta\eta} = \left(\frac{(1-e^{-\gamma})e^{-\Gamma_v}}{\gamma(1-e^{-\Gamma_v})+\Gamma_v e^{-\Gamma_v}}\right)\left[\int_0^{\Delta_v^X} \frac{e^{-\hat{\rho}\tau}}{e^{-\hat{\rho}\Delta_v^X}}\left(\frac{c(\tau)}{c(\Delta_v^X)}\right)^{-\sigma} d\tau + c(\Delta_v^X)\left(\frac{1-e^{-\frac{\hat{\rho}}{\sigma}\Delta_v^C}}{\hat{\rho}/\sigma}\right)\right]$$

The right–hand side is monotonically increasing with Δ_v^X. Given Γ_v, Δ_v^C and $F(\cdot)$,. this condition therefore pins down a unique value of Δ_v^X.

References

Aghion, P., G.M. Angeletos, A. Banerjee, and K. Manova (2005), "Volatility and growth: Credit constraints and productivity–enhancing investment," NBER Working Paper 11349.

Aghion, Philippe, and Giles Saint Paul, "Uncovering some causal relationships between productivity growth and the structure of economic fluctuations: A tentative survey", *Labor*, vol. 12, 1998, 279-303.

Aghion, Philippe and Peter Howitt, "*Endogenous Growth Theory*" Cambridge MIT Press, 1998.

Aguirregabiria, Victor and Cesar Alonso-Borrego, "Occupational Structure, Technological Innovation, and Reorganization of Production" *Labour Economics*, vol. 8 (1), 2001, pp. 43-73.

Baldwin, John, Desmond Beckstead and Guy Gellatly (2004), *Canada's Expenditures on Knowledge Capital*, Statistics Canada.

Barlevy, Gadi (2005a), "Why don't recession encourage more R&D spending?" *Chicago Fed Letter* # 220 (Essays on Issues), Federal Reserve Bank of Chicago.

Barlevy, Gadi (2007), "On the timing of innovation in stochastic Schumpeterian growth models," Federal Reserve Bank of Chicago working paper # WP-2004-11.

Bental, B. and D. Peled (1996), "The accumulation of wealth and the cyclical generation of new technologies: a search theoretic approach," *International Economic Review*, vol. 37, pp. 687–718.

Fatas, Antonio, "Do Business Cycles Cast Long Shadows? Short-Run Persistence and Economic Growth" *Journal of Economic Growth*, vol. 5 (2), 2000, pp. 147-62.

Fay, Jon-A. and James-L. Medoff, "Labor and Output over the Business Cycle: Some Direct Evidence" *American Economic Review,* vol. 75 (4), 1995, pp. 638-55.

Francois, P. and H. Lloyd-Ellis (2003) "Animal Spirits Through Creative Destruction," *American Economic Review*, vol. 93 (3), pp. 530-550

Francois, P. and H. Lloyd-Ellis (forthcoming) "Implementation Cycles, Investment and Growth," forthcoming *International Economic Review*.

Francois, P. and H. Lloyd-Ellis (2007) "Schumpeterian Restructuring," mimeo, Queen's University.

Freeman, Scott, Dong Pyo Hong and Dan Peled, "Endogenous Cycles and Growth with Indivisible Technological Developments," *Review of Economic Dynamics*, vol. 2, 1999, 403–432.

Geroski, P.A. and C.F. Walters, "Innovative Activity over the Business Cycle" *Economic Journal*, vol. 105 (431), 1995, pp. 916-28.

Graham, S. J. H. (2004) "Patenting in the Shadow of Secrecy: Innovators' Uses of U.S. Patent Office Continuation Practice, 1975–2002." In *Continuation, Complementarity, and Capturing Value: Three Studies Exploring Firms' Complementary Uses of Appropriability Mechanisms in*

Technological Innovation. Ph.D. dissertation, University of California, Berkeley.

Graham, S.J.H. and Mowery (2004). "Submarines in Software? Continuations in US Software Patenting in the 1980s and 1990s." *Economics of Innovation and New Technology*, 13:5, pp. 443–56.

Griliches, Z. (1990), "Patent statistics as economic indicators: A Survey," *Journal of Economic Literature*, vol. 28 (4), December, pp. 1651–1707.

Grossman, Gene and Elhanan Helpman, *Innovation and Growth in the Global Economy*, Cambridge, MA.: MIT press, 1991.

Hansen, G. (1985) "Indivisible Labor and the Business Cycle," *Journal of Moneary Economics*, November, pp. 309-327.

Harashima, Taiji (2006), "The Pro–Cyclial R&D Puzzle: Technology Shocks and Pro-Cyclical R&D Expenditure," mimeo, University of Tsukuba.

Kamin, J.Y., I. Bijaoul and R. Horesh (1982), "Some determinants of cost distributions in the process of technological innovation," *Research Policy*, vol. 11, pp. 83–94.

Lemley, M. and C. Shapiro (2005), "Probabilist Patents," *Journal of Economic Perspectives*, 19(2): 75–98.

Matsuyama, Kiminori, "Growing Through Cycles" *Econometrica,* 67 (2), 1999, pp. 335–47.

Matsuyama, Kiminori, (2001) "Growing Through Cycles in an Infinitely Lived Agent Economy," *Journal of Economic Theory*, 100 (October 2001): 220-234.

Nickell, Stephen, Daphne Nicolitsas, Malcolm Patterson, "Does Doing Badly Encourage Management Innovation?" *Oxford Bulletin of Economics and Statistics*, 63 (1), 2001, pp. 5-28.

Paterson, Bryan (2006), "Synergies, Spillovers and the Role of Implementation in the Innovation Process," mimeo, Queen's University.

Rios-Rull, V. and Santaeulalia-Llopis, "Redistributive Shocks and Productivity Shocks," mimeo, University of Pennsylvania, 2006.

Schumpeter, J., "The Explanation of the Business Cycle" *Economica,* 1927, pp. 286–311.

Schumpeter, Joseph, *Capitalism, Socialism and Democracy*, 3rd edition, New York, 1950.

Shleifer, A. (1986), "Implementation Cycles," *Journal of Political Economy*, vol. 94 (6), pp. 1163–1190.

Wälde, Klaus (2005), "Endogenous Growth Cycles," *International Economic Review*, vol. 46 (3), pp. 867-894.

Wälde, Klaus and Ulrich Woitek (2004), "R&D expenditure in G7 countries and the implications for endogenous fluctuations and growth," *Economics Letters*, vol. 82, pp. 91–97.

Appendix B: Stationary Cyclical Equilibrium Growth Path in the Generalized Model

Proposition A1: *The stationary cyclical equilibrium growth path in the generalized model is fully described by the vector* $(\Gamma, \Delta^X, \Delta^C, \hat{z}, \hat{S})$ *and a recurring expansion path for consumption* $\{\hat{c}(t)\}_{\tau=0}^{\Delta^X}$ *which satisfy the following system:*

$$\hat{c}(\tau) = F(\tau,\ e^{-\Gamma+\rho\Delta^X}\hat{c}(\Delta^X),\ e^{-\Gamma}\hat{z}) \qquad \forall\, \tau \in [0, \Delta^X] \tag{1}$$

$$\Gamma = \delta\gamma\Delta^C - \delta\gamma\left(\frac{b}{1-\alpha}\right)^{\frac{1-\alpha}{\alpha}} c\left(\Delta_v^X\right)^{\frac{1}{\alpha}}\left(\frac{1-e^{-\rho\Delta^C}}{\rho}\right) \tag{2}$$

$$\frac{(1-\eta)\mu e^{-\gamma}}{\eta\delta\left(\frac{bc(\Delta^X)}{1-\alpha}\right)^{\frac{1-\alpha}{\alpha}}}\left(\frac{\Gamma}{\gamma\hat{S}(1-e^{-\Gamma-[(1-\alpha)\rho+\phi]\Delta^C}) + \Gamma e^{-\Gamma-[(1-\alpha)\rho+\phi]\Delta^C}}\right) = e^{\gamma\hat{S}} = \hat{z} \tag{3}$$

$$\frac{e^{-\gamma}}{\gamma\delta\eta} = \frac{(1-e^{-\gamma})e^{-\rho\left(\Delta^X+\Delta^C\right)}}{\gamma+(\Gamma-\gamma)\,e^{-\rho(\Delta^X+\Delta^C)}}\left[\int_0^{\Delta^X}\frac{e^{-\rho\tau}}{e^{-\rho\Delta^X}}\left(\frac{c(\tau)}{c(\Delta^X)}\right)^{-\frac{1}{\alpha}}d\tau + (\frac{b}{1-\alpha})^{\frac{1-\alpha}{\alpha}}c(\Delta^X)^{\frac{1}{\alpha}}\left(\frac{1-e^{-\rho\Delta^C}}{\rho}\right)\right] \tag{4}$$

$$\Gamma = \rho\Delta^X + \alpha\rho\Delta^C \tag{5}$$

Proof: The derivations follow the same general steps as in the basic version. A key difference is that in this generalized model the level of technology now pins down the unit cost of final goods production, rather than the skilled wage:

$$w(t)^\alpha w_p(t)^{1-\alpha} = \alpha^\alpha(1-\alpha)^{1-\alpha}e^{-\alpha\gamma}e^{\phi t}\bar{A}_{v-1}, \tag{6}$$

It follows that during the cycle the growth in the wages of workers used in producing intermediate services and those used in final production must be related according to

$$\alpha\frac{\dot{w}}{w} + (1-\alpha)\frac{\dot{w}_p}{w_p} = \phi$$

A second key implication is that the household‘s labour supply condition is tied to the the marginal product of labor according to

$$w_p(t) = bC(t) = (1-\alpha)\frac{Y(t)}{L(t)}$$

Aggregate output can now be expressed as

$$Y(t) = e^{\phi t}\bar{A}_{v-1}\left[1 - H(t)\right]^{\alpha} L(t)^{1-\alpha}. \tag{7}$$

In intensive form this is

$$y(t) = \left[1 - H(t)\right]^{\alpha} L(t)^{1-\alpha}. \tag{8}$$

It follows that

$$L(t) = \left(\frac{1-\alpha}{bc(t)}\right)^{\frac{1}{\alpha}}\left[1 - H(t)\right]$$

The household's Euler equation during the cycle can be expressed as

$$\frac{\dot{c}(t)}{c(t)} = r(t) - \rho - \phi, \tag{9}$$

and normalized potential productivity evolves according to

$$\frac{\dot{z}(t)}{z(t)} = \frac{\dot{Z}}{Z} - \phi = \mu\gamma\left(\frac{\left[1 - H(t)\right]\left(\frac{1-\alpha}{bc(t)}\right)^{\frac{1-\alpha}{\alpha}} - c(t)}{z(t)}\right), \tag{10}$$

The Expansion : Combing these conditions in the same way as before yields the dynamical system:

$$\frac{\dot{c}(t)}{c(t)} = \mu\gamma\left(\frac{\left(\frac{1-\alpha}{bc(t)}\right)^{\frac{1-\alpha}{\alpha}} - c(t)}{z(t)}\right) - \rho \tag{11}$$

$$\frac{\dot{z}(t)}{z(t)} = \mu\gamma\left(\frac{\left(\frac{1-\alpha}{bc(t)}\right)^{\frac{1-\alpha}{\alpha}} - c(t)}{z(t)}\right). \tag{12}$$

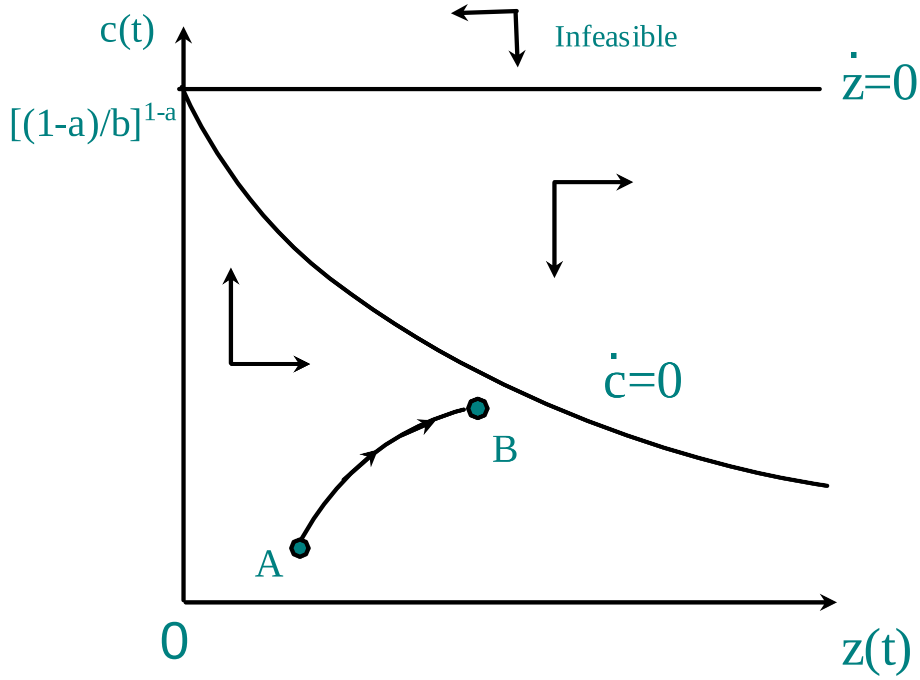
These dynamics are illustrated in the phase diagram below. As can be seen, these dynamics are much the same as without production labor.

The Contraction: During this phase, $H(t) > 0$ and $w(t) = \delta\eta V^{D}(t)$. It follows that the interest rate is given by

$$r(t) = \frac{\dot{V}^{D}(t)}{V^{D}(t)} = \frac{\dot{w}}{w} = \frac{\phi}{\alpha} - \left(\frac{1-\alpha}{\alpha}\right)\frac{\dot{w}_{p}}{w_{p}} = \frac{\phi}{\alpha} - \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{\dot{c}}{c} + \phi\right) = \phi - \left(\frac{1-\alpha}{\alpha}\right)\frac{\dot{c}}{c}. \tag{13}$$

where the last equality uses the household labour supply condition. Using the households Euler equation it follows that

$$\frac{\dot{c}(t)}{c(t)} = -\alpha\rho \tag{14}$$



and

$$r(t) = (1-\alpha)\rho + \phi \tag{15}$$

The initial decline in output due to the fall in R&D investment demand must be proportional to the fraction of labor hours withdrawn from production:

$$H_v = 1 - \left(\frac{b}{1-\alpha}\right)^{\frac{1-\alpha}{\alpha}} c\,(T_v^*)^{\frac{1}{\alpha}} \tag{16}$$

Since R&D ceases, all output is used for consumption, $c(t) = y(t)$. Equating, implies that total labour allocated to production falls to a constant level

$$L(t) = \frac{1-\alpha}{b} \qquad \forall\ t \in (T_v^*, T_v) \tag{17}$$

and

$$c(t) = y(t) = (1 - H(t))^{\alpha} \left(\frac{1-\alpha}{b}\right)^{1-\alpha} \tag{18}$$

The Implementation Boom: The household's Euler equation implies that

$$R_0(T_v) - R_0(T_{v-1}) = \rho\Delta_v + \Gamma_v + \phi\Delta_v \tag{19}$$

During the expansion, the discount factor must grow by $\ln \frac{Z(T_v^*)}{Z(T_{v-1})} = -\phi\Delta^C + \ln \frac{Z(T_v)}{Z(T_{v-1})}$. During the downturn the interest rate is $\phi + (1-\alpha)\rho$. Combining these facts with (19), it follows that across the boom the discount factor must satisfy

$$\beta(T_v) = \rho\Delta_v + \Gamma_v + \phi\Delta_v - \ln\frac{Z(T_v)}{Z(T_{v-1})} - (1-\alpha)\rho\Delta_v^C \tag{20}$$

In the stationary equilibrium $\ln\frac{Z(T_v)}{Z(T_{v-1})} = \Gamma_v + \phi\Delta_v$ and, following similar logic to that used before, the discount factor across the boom is equal to productivity growth, $\beta(T_v) = \Gamma_v$. It follows that

$$\Gamma_v = \rho\Delta_v^X + \alpha\rho\Delta_v^C \tag{21}$$

Since the interest rate is $\phi + (1-\alpha)\rho$ through the downturn it is also the case that $\beta(T_v^*) = (\phi + (1-\alpha)\rho)\Delta^C + \Gamma_v$ and that, using the household Euler equation:

$$\rho\Delta_v^X - \Gamma_v + \ln c(T_v^*) = \ln c_0(T_v), \tag{22}$$

where $\Delta_v^X \equiv T_v^* - T_{v-1}$ denotes the expansion length.

No-arbitrage in R&D: As before, we can express the value of a new idea at the peak of the cycle as

$$\Omega(T_v^*) = \frac{(1-\eta)w(T_v^*)}{\eta\delta}\left(\frac{S(T_v^*) - S(T_v)}{S(T_v^*) - S(T_v)e^{-\beta(T_v^*)}}\right) \tag{23}$$

But, in this case

$$w(T_v^*)^\alpha = \frac{\alpha^\alpha(1-\alpha)^{1-\alpha}e^{-\alpha\gamma}e^{\phi T_v^*}\bar{A}_{v-1}}{w_p(T_v^*)^{1-\alpha}} \tag{24}$$

$$w(T_v^*) = \frac{\alpha e^{-\gamma}e^{\phi T_v^*}\bar{A}_{v-1}}{\left(\frac{bc(T_v^*)}{1-\alpha}\right)^{\frac{1-\alpha}{\alpha}}} \tag{25}$$

Following the same steps as in Proposition 1, we get

$$\frac{(1-\eta)\mu e^{-\gamma}}{\eta\delta\left(\frac{bc(T_v^*)}{1-\alpha}\right)^{\frac{1-\alpha}{\alpha}}}\left(\frac{\Gamma_v}{\gamma S(T_v^*) - (\gamma S(T_v^*) - \Gamma_v)e^{-\beta(T_v^*)}}\right) = e^{\gamma S(T_v^*)}. \tag{26}$$

No-arbitrage in commercialization: As before, the discounted monopoly profits from owning an innovation at time T_{v-1} is given by

$$\begin{aligned} V_0^I(T_{v-1}) = {} & \alpha(1-e^{-\gamma})Y_0(T_{v-1})\left[\int_{T_{v-1}}^{T_v^*} e^{-\int_{T_{v-1}}^{\tau}[r(s)-g(s)]ds}d\tau\right. \\ & \left. + e^{-\int_{T_{v-1}}^{T_v^*}[r(s)-g(s)]ds}(1-H_v)\int_{T_v^*}^{T_v} e^{-\int_{T_v^*}^{\tau}[r(s)-g(s)]ds}d\tau\right] + P(T_v)e^{-\beta(T_{v-1})}V_0^I(T_v) \end{aligned} \tag{27}$$

Substituting in the relevant values derived above and integrating, we get

$$\begin{aligned} V_0^I(T_{v-1}) &= \alpha(1-e^{-\gamma})Y_0(T_{v-1})\left[\int_{T_{v-1}}^{T_v^*} e^{-\rho(\tau-T_{v-1})}\left(\frac{c(\tau)}{c_0(T_{v-1})}\right)^{-\frac{1}{\alpha}} d\tau \right. \\ & \left. +e^{-\rho\Delta_v^X}\,(c_0(T_{v-1}))^{\frac{1}{\alpha}}\,(\frac{b}{1-\alpha})^{\frac{1-\alpha}{\alpha}}\left(\frac{1-e^{-\rho\Delta_v^C}}{\rho}\right)\right] + P(T_v)e^{-\beta(T_{v-1})}V_0^I(T_v) \end{aligned} \tag{28}$$

Following the same procedure as before, but noting that now $\beta(T_{v-1}) = \alpha(\Gamma_v + \phi\Delta_v) + \rho\Delta_v$, yields

$$\frac{e^{-\gamma}}{\delta\eta} = \left(\frac{(1-e^{-\gamma})c_0(T_{v-1})^{\frac{1}{\alpha}}}{1-(1-\Gamma_v/\gamma)e^{-\rho\Delta_v}}\right)\left[\int_{T_{v-1}}^{T_v^*} e^{-\rho(\tau-T_{v-1})}c(\tau)^{-\frac{1}{\alpha}}d\tau + e^{-\rho\Delta_v^X}(\frac{b}{1-\alpha})^{\frac{1-\alpha}{\alpha}}\left(\frac{1-e^{-\rho\Delta_v^C}}{\rho}\right)\right] \tag{29}$$

Using (22), we can write this as

$$\frac{e^{-\gamma}}{\gamma\delta\eta} = \left(\frac{(1-e^{-\gamma})\left[e^{\rho\Delta_v^X-\Gamma_v}c(T_v^*)\right]^{\frac{1}{\alpha}}}{\gamma-(\gamma-\Gamma_v)e^{-\rho\Delta_v}}\right)\left[\int_{T_{v-1}}^{T_v^*} e^{-\rho(\tau-T_{v-1})}c(\tau)^{-\frac{1}{\alpha}}d\tau + e^{-\rho\Delta_v^X}(\frac{b}{1-\alpha})^{\frac{1-\alpha}{\alpha}}\left(\frac{1-e^{-\rho\Delta_v^C}}{\rho}\right)\right] \tag{30}$$

After some further manipulation we get

$$\frac{e^{-\gamma}}{\gamma\delta\eta} = \left(\frac{(1-e^{-\gamma})e^{-\rho\Delta_v}c(\Delta_v^X)^{\frac{1}{\alpha}}}{\gamma(1-e^{-\rho\Delta_v})+\Gamma_v e^{-\rho\Delta_v}}\right)\left[\int_0^{\Delta_v^X} \frac{e^{-\rho\tau}}{e^{-\rho\Delta_v^X}}c(\tau)^{-\frac{1}{\alpha}}d\tau + (\frac{b}{1-\alpha})^{\frac{1-\alpha}{\alpha}}\left(\frac{1-e^{-\rho\Delta_v^C}}{\rho}\right)\right] \tag{31}$$